Foundational Concepts Underlying a Formal Mathematical Basis for Systems Science

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Abstract

We propose Category Theory as the formal, mathematical language and theory for studying systems. But, to understand the applicability of Category Theory to Systems Science we must go back to certain prior foundational concepts which, first, must be clearly elucidated and their contexts made explicit.

Keywords: science, formal systems, category theory.

Science is a Formal System for Developing Valid Knowledge

We propose Category Theory as the formal mathematical language for a science of studying systems – all kinds of systems – from the quantum to cosmological, from the social to technological, and beyond. In the same way that Category Theory has been used to study all of mathematics, Category Theory provides a coherently consistent means for studying the systems of science. This approach may lead us to new understandings in the paradigms of science. At present, science is often misunderstood (Pollack, 2003), and there are problems in adequately communicating its concepts without inconsistency, incompleteness or paradoxes. Part of that problem is incompleteness internal to the concepts of science, while another part of the problem manifests externally in communication between elements within the paradigm – specifically, in the way science interacts between our worlds - plural. In communication, Eliezer Yudkowsky of the Machine Intelligence Institute cautions us about expecting short inferential distances (Yudkowsky, 2007), "the legendary difficulty most scientists have in communicating with a lay audience - or even communicating with scientists from other disciplines". Therefore, inferential distance is the number of necessary transformative steps from a general prior knowledge to a new, adequate understanding of knowledge concepts.

Category Theory and its mathematical language exhibit extremely long inferential distances between its abstract concepts and our general background knowledge. The necessary inferences have historically been beyond the experiences of most people including systems scientists and systems engineers. This holds true even for some mathematicians, which is why Category Theory was once considered "abstract nonsense" shortly after it was introduced. Category Theory is a formal language, which means that because it is consistent within its context, it is also incomplete. Why, then, would we propose Category Theory as a formal foundational approach? If we use a system of mutually-connected contexts at different levels of abstraction, it allows us to ameliorate potential incompleteness, inconsistency and paradoxes found in other methods and paradigms. This allows Category Theory to be a proven, powerful and expressive formal language for transforming, verifying and validating prior beliefs into dependable knowledge. It does so with a coherent consistency that natural language has never achieved.

Used in conjunction with myriad other theories – such as Inverse Theory – Category Theory allows scientists to assimilate new information onto the priors in evolving and developing valid knowledge, starting with, but at least partially independent from, human belief. This is precisely how we have characterized science at the beginning of this paper.

> “The beauty of category theory is that it allows one to be completely precise about otherwise informal concepts. Abstract algebraic notions such as free constructions, universality, naturality, adjointness, and duality have precise formulations in the theory. Many algebraic constructions become exceedingly elegant at this level of abstraction." -- Kozen et al. in *Automating Proofs in Category Theory* (Kozen, Kreitz, & Richter, 2006).

A *Category* in Category Theory can be used to precisely represent any system of interest, its objects, structure, behavior and morphism. Physicist John Baez and mathematician Mike Stay even refer to Category Theory as “A Rosetta Stone” (Baez & Stay, 2011). A categorical description can encapsulate all of a systems potential dynamics when it exists in non-equilibrium within its contexts, and non-equilibrium systems are an important difference in this paradigm of science. Specifically, non-equilibrium dynamics are different from the dominant static method of modeling systems at equality which can easily be seen in our prior mathematics.

In this way, a new dynamical paradigm emerges where categorical descriptions hold for simple systems to systems of extraordinary complexity at various distances from equilibrium. Category Theory becomes a coherently consistent indirect referential language to all systems including those of science itself.

What is missing in understanding Category Theory are models and contexts of the foundational concepts necessary to make the language of Category Theory “reachable” and understandable to the systems scientist. These concepts exist concurrently at many levels of abstraction – not just as mental notions.

We start with a few axiomatic concepts. Axioms, as characterized by category theorist Paul Taylor, are “not something obviously true, but serve as starting points for reasoning toward theories” (Taylor, 2011). These axioms can be understood as those abstract concepts, including mental notions and other abstract forms. Since many reading this paper are familiar with the Unified Modeling Language, we will utilize that language as well as a natural language, American English, to speak of these concepts. Axioms are composed – built upon and refined into larger wholes. It is not by random chance that the mathematical language of Category Theory allows us to start from prior foundational concepts, build up, describe, develop and refine our theories about them.

An Object

Our first axiom is that of an object. In a natural language, objects are usually represented by nouns. An object is any existential thing consisting of an entity, an identity, the ability to aggregate other objects as components, and the necessity of a context – the frame of reference - in which it exists. When we refer to an object’s entity, we directly refer to it by its identity – its

"thisness" due to haecceity. Whenever we refer to "this" thing in a particular frame of reference, we are referring to an object through its identity. An object's identity remains constant throughout the lifetime of the object, whereas an object's entity may age, grow, die or disappear from its context. Simultaneously, when we make reference to an object's identity, we also indirectly refer to all the identities of concepts that are connected beyond that context.
One example is from our prior mathematics, an unordered set is simply an object comprised of other objects – like the gallimaufry of things (the aggregation) thrown into a junk drawer (identified as the object).

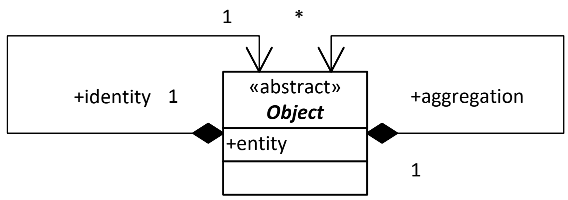

Figure 1 - An Object

Notice that Figure 1, (which actually represents the class of objects – a kind of category) exhibits a structure formed by the arrows in an ever increasing cycle of abstraction (the context). It does present a "which came first – chicken or egg" – conundrum. Since both chickens and eggs can be identified as objects, we chose an object as our primary axiom. The founders of Category Theory – Samuel Eilenberg and Saunders Mac Lane - thought objects played a secondary role (Eilenberg & Mac Lane, 1942), and have started with categories, for very interesting reasons we will only now hint at by the term "abstract".

Objects need not exclusively be physical objects, however. It can be "this" notion, intuition, thought or idea (mental), or "this" triangular shape, directed graph or network form (platonic form). It may be "this" number, algebraic variable or equation in mathematics, "this" perception, "this" schema, or "this" experiment. An object is the least specific (therefore, most general) term of an identified thing existing within its context. The philosopher Charles Sanders Peirce even proposed that people could be referred to by their identities as objects.

Functions and Functional Objects

A function is an abstract description of some behavior. In our natural language, functions are usually represented by verbs, such as the word "evolve" or "order". For a specific example, when we think of the function "run" we realize that dogs run, computer programs run, refrigerators run, in the long run time runs. Functions, in and of themselves, are not objects however, but they can resemble aspects – cross-cutting concerns.

It is easier for us to conceptualize the abstraction of a function when we combine it with an identifiable object we can actually reference – a Functional Object. These are usually represented by a phrase or sentence – consisting of a noun–verb pair. Mathematically, we can think of addition or subtraction, even equality, morphism or being as functional objects identified and represented by graphical symbols. But, functions and functional objects go far beyond mere mathematical concepts.

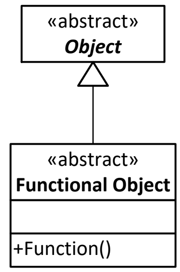

Figure 2- Functional Object w/ Function

Morphism

A morphism is a function (and since it is identifiable, a functional object) that represents some transformation in the structural form or change in the number or nature of the objects in the aggregation. In this regard, it is correlated if not actually causal with transformation or change, usually in some finite period of time. In many languages – including Category Theory, the UML or SysML - a morphism is represented by some form of an *arrow* symbol that maps the prior (this) to the result (that).

Figure 3 - A Morphism

Category

Here is our problem: Most people when they think of categories conceive of classification systems – hierarchies of classes, taxonomies and ontologies. This is problematic because those kinds of categories are far more sophisticated and refined – more concrete - concepts than are necessary or useful as a category at this point.

A category is simply an *object* that adds or imposes structure upon its aggregation of objects. That’s it. In its abstract form, it doesn’t say specifically what that structure is or how that category structures its aggregation – but it seems like functions, functional objects or morphisms might seem reasonable suspects associated with structure. Do you ever wonder how we can ever identify one object from another? This concept of structure becomes quite interesting – quickly and deeply.

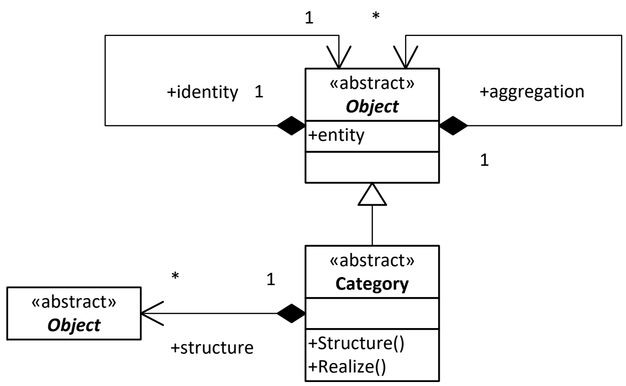

Figure 4- A Category

Since a category is an object, the aggregation of objects can include categories in that aggregation.

In the UML-like diagrams Figures 1, 2, 3 and 4, the context is described by the stereotype <<abstract>>. Notice that the class "Object" is shown as *abstract* as well. In the supremely abstract conceptual context (world) all concepts are realized as categories, but referenced by their abstract object's identity. That means any category, at any level of abstraction, can be identified as "this" category, distinct from all other categories.

It is now time to introduce a new, perhaps foreign, concept – and a new term, one that is used in the language of category theory – a *functor*. It is based upon a composition of the few foundational concepts we have already presented.

Functors

Functors are Categories – structures whose aggregation of objects are Functional Objects or Categories of Functional Objects (and so on, ad infinitum). It makes logical sense therefore, that – since a functor is a category – the aggregation of objects of functors can also include functors. One common example of a functor is its analog in Functional Computer Programming – perhaps a lambda expression – which is a *very* specific example of the concept of a functor. A routine or subroutine in Structured Programming, or a method in Object Oriented programming, or an aspect in Aspect Oriented programming are also Functors. Any complex composition of verbs or adverbs in natural languages may also be considered functors.

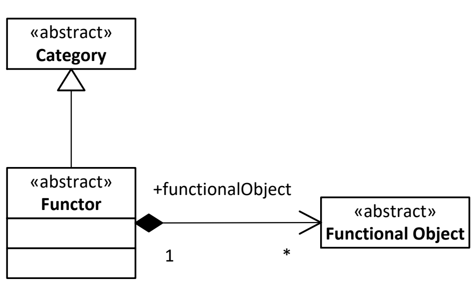

Figure 5- A Functor

In Category Theory, the theoretical and mathematical constructs derived from this simple abstraction of a functor – and its profound relationship with science and with systems – will serve as construction templates for models, including models of theories, models of systems, and even models of the paradigms of science.

Contexts

A Context is a Category realized from a higher level of abstraction whose structure conditions and creates the possibility space for the existence of its aggregation of objects. It may be said that a context exists in duality, internally, with the objects in its aggregation because it structurally conditions the possibility for their existence. That condition is only probabilistic, since it does not guarantee any actual existence. A context also exists in duality, externally, with its more abstract category from which it inherits these structural capabilities. Here we must clarify how contexts and objects are realized (the dashed arrows below, oversimplified) as a duality from more abstract categories. But also there is duality between contexts and their objects. Objects exist in both an internal and external relationship with concepts that condition them. We refer to these dualities between three elements as a triality, the relationship between more abstract categories and the categorical combination of less abstract objects in contexts.

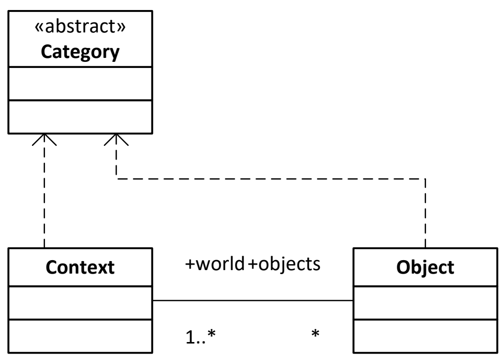

Figure 6 - Triality of Categories, Objects and Contexts

There are perhaps infinitely many contexts in which an object may exist. In science, we often concern ourselves with physical environmental contexts – temperatures, pressures, and gravity – but our considerations must also extend into economic, financial, technological, socio-societal, spatio-temporal, legal, ethical, political and psychological contexts as well – just to name a few. All of these contexts condition the probable existence of the objects within their categories by communication (in the broadest sense of the word) – exchanging matter, energy, information or entropy - between the objects in the aggregation and between the objects and their context.

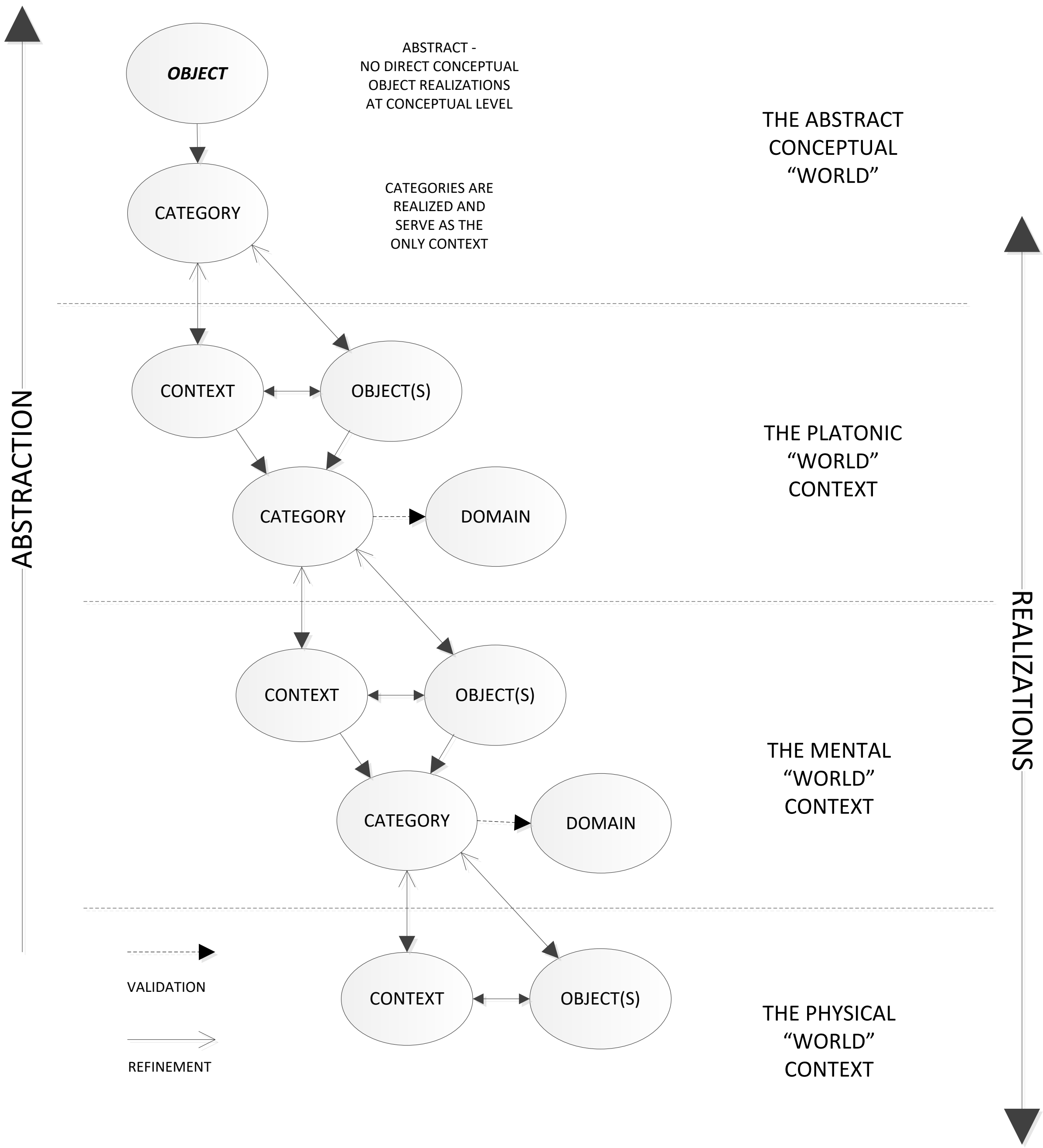

Figure 7- Abstractions and Realizations

In Figure 7, we can see how abstract concepts exist in a superposition with all of the objects, categories, contexts and possible domains from which they are derived. The fancy mathematical term for this series of morphisms is *A Homology of Chain Complexes* due to the similarity of the patterns of its evolution. But, this diagram is incomplete because it is represented as a hierarchy of Worlds solely along a dimension of Abstraction. Science is not strictly hierarchical. It is more complex – and that may prove to be a very good thing.

"Worlds" as Contexts

The term World is used for historical consistency with the philosopher, Karl Popper (Popper, 1978), and the polymath, Roger Penrose (Penrose, 2005). Instead of using the UML we will simplify our notation by using an ellipse for category-like objects, and arrows for morphisms.

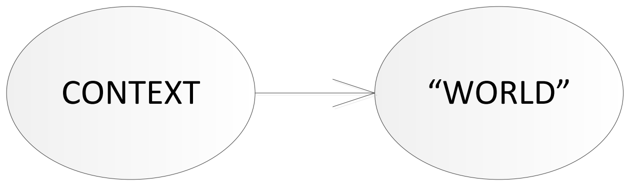

Figure 8- "World" as Contexts

A world is a context, which as we have seen is a category existing at various levels of abstraction whose objects are the many contexts that affect, structure and condition the existence of the objects of that world. Popper identified three worlds developed though the history of philosophy – The Physical World, The Mental World, and a somewhat abstract world, The Products of the Mind. Unfortunately, he named these worlds, World 1, World 2 and World 3 – leaving the conceptualization of "products of the mind" to the reader. Furthermore, no specific structure was identified between Popper's Worlds.

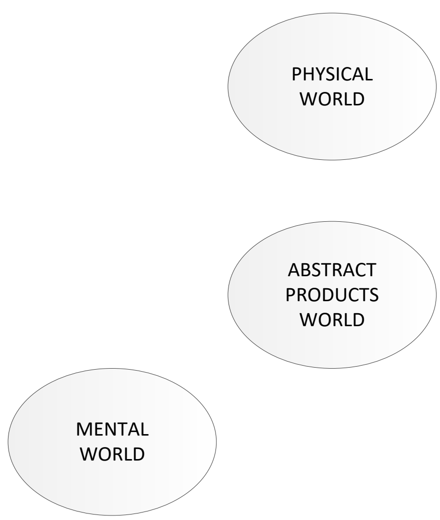

Figure 9- Popper's Three Worlds

Roger Penrose, an avowed Platonist, described his three worlds quite differently from Popper – The Physical World, The Mental World, and the Platonic World of Mathematical Forms. Additionally, Penrose identified three mysteries of representation between these worlds.

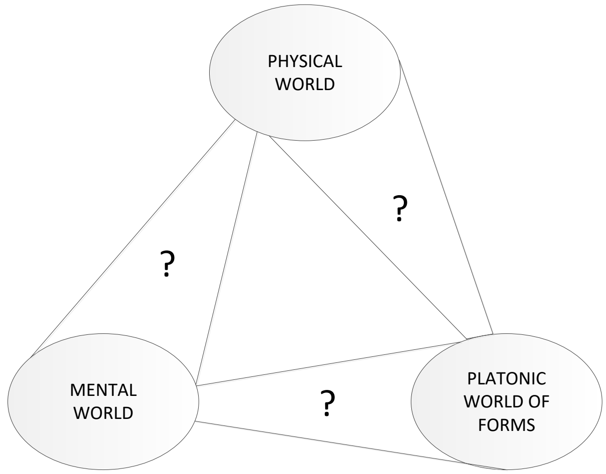

Figure 10- Penrose's Three Worlds & Three Mysteries

The mystery (?) is how only a small part of one world, functioning as a meta-world, can describe almost all the objects, structure, behavior and morphism of the next world.

I have recast Popper's World 3 as an Abstract Conceptual World and added Penrose's three worlds. Instead of Penrose's Three Directional Mysteries, we end up with 6 bi-directional representations – the Interstitial Languages – that mutually couple the Four Worlds.

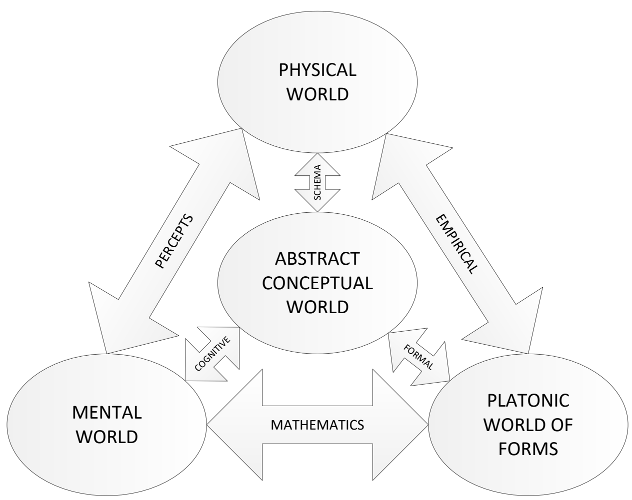

Figure 11- Lloyd's 4 Worlds & 6 Interstitial Languages

We approach completeness by recurrence through the systems formed by the interconnection and mutual interaction of these languages. It is now common to ask: What do we mean by "system"?

System

A system is precisely characterized by a category, being two or more coupled (structured) objects that permit communication (in the broadest sense of the word) – the exchange of matter, energy, information or entropy when not at equilibrium. Furthermore, the structure that permits the coupling at least partially subsumes the identities of the aggregate objects. In a physical sense, the bonding associated with coupling lowers the potential energy within the system.

In Figure 11, there are three coupled systems that form the interactions between worlds.

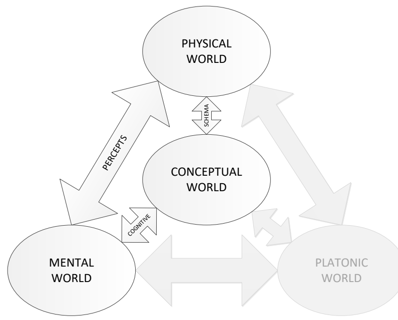

Figure 12- Philosophical System

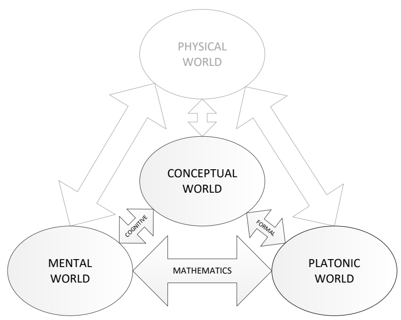

Figure 13- Mathematical System

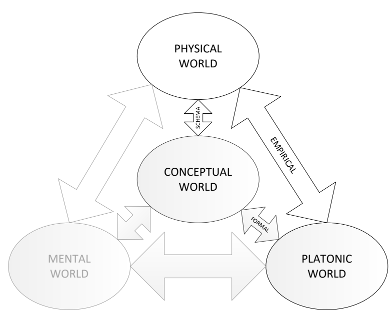

Figure 14- The Empirical System

Theories and Models

Up until this point, we have studiously avoided any direct reference to Category Theory or its mathematical language. But it is obvious that Category Theory is a theory. What may not be obvious is that a theory is an abstract system as described above existing in the context of an Abstract World. There is a recognizable (therefore identifiable) pattern to all theories, and in the language of category theory it is simply this, in its entirety:

$$F: C \rightarrow D$$

Where,
F is a model of that theory – a functor,
C is a category of priors, to be transformed by *F*, and
D is a category of results of the morphism.

William Lawvere labored over the role of a functor as a model of a theory in (Lawvere, 1963, 2004) . While the nature of the functor(s) involved can become more complex – including certain categories of functors called natural transformations – the pattern still holds for the concept of a theory. Symbolically, the letters may change, as may the fonts, used to denote different kinds of categories – groups, spaces, etc. - but the pattern always remains the important point.

In this case, *F* is known as the Forward Model. Often in simple theories, *C* is a category of parameters and *D* is the category of data. But, in general we may substitute any category for *C*, and the resultant category for *D*. Alas, this too, is incomplete. In the introduction to this paper, we mentioned Inverse Theory as a means of refinement to models of prior concepts.

$$F^{OP} : D \rightarrow C$$

Where F^{OP} is the inverse model. The how and why we bring this up is only to peak your interest for further research, but this bi-directionality is a universal aspect in all of science – an universal property called a natural transformation in theories.

We expand on this simple notion of a theory to create a category-theoretic meta-language for all interstitial languages in the interconnected system of worlds. The forward and inverse language models are shown by the bi-directional arrows that realize and refine concepts between these worlds.

Domains

All of the previously presented concepts are extremely simple things built upon simpler things. Domains, unfortunately, are not. Domains are short for "domains of validity" that establish the boundaries and validity for all formal systems. It is also unfortunate that in the literature of category theory, *C* in the equation above is called "the domain", and *D* is called the co-domain. Merely calling a category "a domain" does not make it valid, however. The category-theoretic mechanisms of pullbacks and pushouts, of adjunction and natural transformations, are far beyond the concepts of importance in this paper to establish validity. What is an important concept to understand is that the challenge in all of science is to transform the categorical worlds – their contexts, objects, structure, behavior and morphism – into Domains of Validity.

Paradigms

A scientific paradigm (Kuhn, 1962, 1996) is the final concept in a category-theoretic approach to a science of systems. A paradigm is a category of complex-coupled Domains – we present as a framework – that operates at some distance near equilibrium. What emerges from this interaction is valid knowledge that refines and delineates our belief. The actual path through the framework is probably not known a priori due to the recursive nature of possible paths. There may be many potentially valid paradigms of science, but all of the homological chain of objects and transformations in the superposition underlying that framework must also be valid. Which is the best fitting, most plausible and likely paradigm can be probabilistically refined by Inverse Theory.

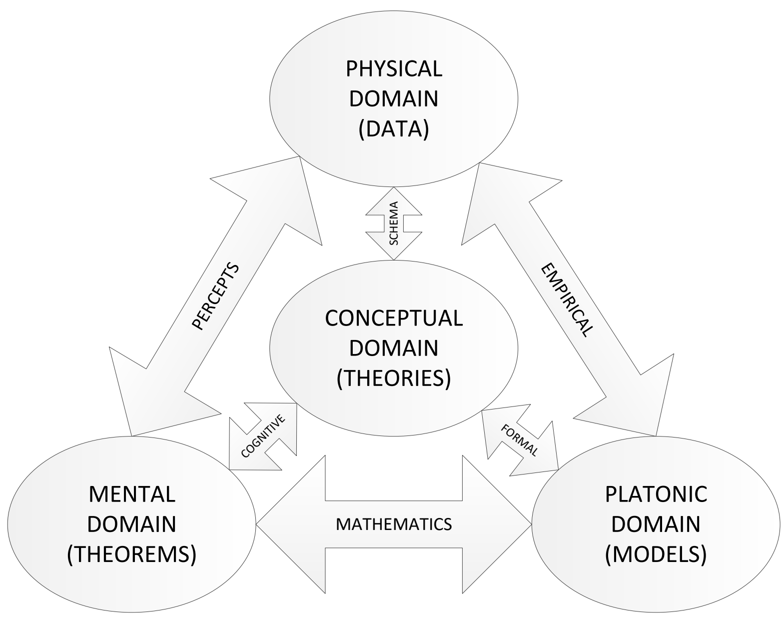

Figure 15- Foundational Framework for a Science of Systems (Lloyd)

Figure 15 represents a paradigm for a science of systems as a framework of complex-coupled categorical, contextual worlds that have been formally transformed into domains of validity. While the figure is shown in the static medium of this printed page, the dynamic, mutual interactions in and between these worlds when not-at-equilibrium – a phenomenon called systematicity – result in an emergent property: The development of valid knowledge which can be quite different from human belief.

The mechanism of this transformation is beyond the scope of this paper, but can be modeled, simulated and visualized by computer programs extended from the formalism. Likewise, the concepts common to these worlds can be verified and validated. At this point in our history, the only known path to discovering truth and true knowledge is predicated on the convergence of valid prior foundational concepts through the dynamical system of the framework.

Summary

We have proposed category theory and its mathematical language as the foundation the systems of science – and therefore the Science of Systems. Because the formal language of category theory exhibits extremely long inferential distances through a large number of transformations called a homology of chain complexes, many people have difficulties understanding how that language is applicable to science – or, indeed, anything else. In an effort to make category theory easier to understand, we have built up categorical contexts starting from simple, primitive concepts. From these concepts, we have formalized the many transformations at work between contexts at different levels of abstraction – contexts we call worlds that are mutually meta-connected to each other in a way that unifies Popper and Penrose.

We have shown in great generality how verification and validation of concepts all along that complex chain become a necessary precursor in developing valid knowledge. In the process, we have also shown how the paradigms of science become the larger context for making science easier to understand as well. Science is no longer merely a philosophical discipline based upon justified true belief, but a somewhat balanced, recurrent, formal dynamical system of philosophical, mathematical and empirical refinements by non-equilibrium conditions that challenge our belief, and from which valid knowledge emerges.

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