Eigenform - An Introduction
by Louis H. Kauffman

Abstract: This essay is a discussion of the concept of eigenform, wherein an object is seen to be a token for those behaviours that lend it its apparent stability . Idealization arises naturally in the from of the creation of such tokens.

I. Introduction
This essay is a explication of the notion of eigenform as pioneered by Heinz von Foerster in his paper [5] and explored in papers of the author [11, 12]. In [5] Heinz performed the magic of convincing us that the familiar objects of our existence can be seen to be nothing more than tokens for the behaviors of the organism that create stable forms. This is not to deny an underlying reality that is the source of these objects, but rather to emphasize the role of process and the role of the organism in the production of a living map that is so sensitive that map and territory are conjoined. Von Foerster's papers [5,6,7] in the book [4] were instrumental in pioneering the field of second order cybernetics.

"I am the observed link between myself and observing myself." [6]

Such an attitude toward objects makes it impossible to discriminate between the object as an element of a world and the object as a token or symbol. If, in this way, we take appearance for reality, then there cannot be any essential difference between the world and the language (in a generalized sense of language) that "describes" it.

The notion of an eigenform is inextricably linked with second order cybernetics. One starts on the road to such a concept as soon as one begins to consider a pattern of patterns, the form of form or the cybernetics of cybernetics. Such concepts appear to loop around upon themselves, and at the same time they lead outward to new points of view. Such circularities suggest a possibility of transcending the boundaries of a system within. When the circular concept is called into being, the boundaries turn inside out.

We take on the possibility that there are no objects separate from our actions. The apparent solidity of external forms is a mirror of the stability (such as it is) of the process by which these forms come into being.

Forms are created from the concatenation of operations upon themselves and objects are not objects at all, but rather indications of processes.

An object , in itself , is a symbolic entity, participating in a network of interactions, taking on its apparent solidity and stability from these interactions. We ourselves are such objects, we as human beings are "signs for ourselves", a concept originally due to the American philosopher C. S. Peirce [10]. Eigenforms are mathematical companions to Peirce's work.

Von Foerster performed a creative act that invites each of us into an unending epistemological investigation. The key to this act is the stance of an observing system. In an observing system, what is observed is not distinct from the system itself, nor can one make a separation between the observer and the observed. The observer and the observed stand together in a coalescence of perception. From the stance of the observing system all objects are non-local, depending upon the prescence of the system as a whole. It is within that paradigm that these models begin to live, act and converse with us. We are the models. Map and territory are conjoined.

II. Objects as Tokens for Eigenbehaviours

In his paper "Objects as Tokens for Eigenbehaviours" [5] von Foerster suggests that we think seriously about the mathematical structure behind the constructivist doctrine that *perceived worlds are worlds created by the observer.* At first glance such a statement appears to be nothing more than solipsism. At second glance, the statement appears to be a tautology, for who else can create the rich subjectivity of the immediate impression of the senses? At third glance, something more is needed. In that paper he suggests that the familiar objects of our experience are the fixed points of operators. These operators *are* the structure of our perception. To the extent that the operators are shared, there is no solipsism in this point of view. It is the beginning of a mathematics of second order cybernetics.

Where are these operators and where are their fixed points?
Lets start back closer to the beginning. Wittgenstein says, at the begiinng of the Tractatus [17],
"The world is everything that is the case."

What is the case is the idea of distinction, including the idea that there is a world. It is tempting to succumb to the idea that behind this tapestry of distinction there is a hidden inner mechanism of the "thing in itself" hiding behind a world of appearances. That "thing in itself" is the other side of the distinction that delineates a world of appearances. One can take the point of view that the perceived world is the world of appearances. But one can take the agnostic point of view that a distinction can be deeply investigated from one of its sides without a belief in the existence of an unobservable side. It is, I believe, this agnostic point of view that leads directly to objects as tokens for eigenbehaviours.

For consider the relationship between an observer O and an "object" A. The key point about the observer and the object is that "the object remains in constant form with respect to the observer". This constancy of form does not preclude motion or change of shape. Form is more malleable than the geometry of Euclid. In fact, ultimately the form of an "object" is the form of the distinction that "it" makes in the space of our perception. In any attempt to speak absolutely about the nature of form we take the form of distinction for the form. (parphrasing Spencer-Brown [3]). It is the form of distinction that remains constant and produces an apparent object for the observer. How can you write an equation for this? The simplest route is to write

$$O(A) = A.$$

The object A is a fixed point for the observer O. The object is an eigenform. We must emphasize that this is the most schematic possible description of the condition of the observer in relation to an object A. We only record that the observer as an actor (operator) manages through his acting to leave the (form of) the object unchanged. This can be a recognition of the symmetry of the object but it also can be a description of how the observer, searching for an object, makes that object up (like a good fairy tale) from the very ingredients that are the observer herself. This is the situation that Heinz von Foerster has been most interested in studying. As he puts it, if you give a person an undecidable problem, then the answer that he gives you is a description of himself. And so, by working on hard and undecidable problems we go deeply into the discovery of who we really are. All this is symbolized in the little equation $O(A) = A$.

And what about this matter of the object as a token for eigenbehaviour? This is the crucial step. We forget about the object and focus on the observer. We atttempt to "solve" the equation $O(A) = A$ with A as the unknown. Not only do we admit that the "inner" structure of the object is unknown, we adhere to whatever knowledge we have of the observer and attempt to find what such an observer could observe based upon that structure.

III. Objects

What is an object? At first glance, the question seems perfectly obvious. An object is, well ... An object is a thing, a something that you can pick up and move and manipulate in three dimensional space. An object is three dimensional, palpable, like an apple or a chair, or a pencil or a cup. An object is the simplest sort of thing that can be subjected to reference. All language courses first deal with simple objects like pens and tables. La plume est sur la table.

An object is separate from me. It is "out there". It is part of the reality separate from me. Objects are composed of objects, their parts. My car is made of parts. The chair is a buzzing whirl of molecules. Each molecule is a whirl of atoms. Each atom a little solar system of electrons, neutrons and protons. But wait! The nucleus of the atom is composed of strange objects called quarks. No one can see them. They do not exist as separate entities. The electrons in the atom are special objects that are not separate from each other and from everything else. And yet when you observe the electrons, they have definite locations.

The physicist's world divides into quantum objects that are subject to the constraints of the uncertainty principle, and classical objects that live in the dream of objective existence, carrying all their properties with them. The difference between the quantum level of objects and the classical level of objects is actually not sharp. From the point of view of the physicist all phenomena are quantum phenomena, but in certain ranges, such as the world of the very small, the quantum effects dominate. It is not the purpose of this essay to detail this correspondence, but a little more information can be found in section 8 of this essay.

A classical object has a location at a given time. You can tell where it is. You can tell a story of where it has been. If the classical object breaks up into parts, you will be able to keep track of all the parts.

Yet electrons and positrons can meet each other and disappear into pure energy! Should we allow objects to disappear? What sort of an object is the electromagnetic field of radio and television signals that floods this room?

Is my thought to be thought of as an object? Can I objectify my thought by writing it down on paper or in the computer? Am I myself an object? Is my body an object in the three dimensional space? Is the space itself an object? Objects have shape. What is the shape of space? What is the shape of the physical universe. What is the shape of the Platonic universe?

It seemed simple. Then, with more experience, the transformations of pattern that formed the space and the objects in it began to appear highly interwoven. In the physical microworld, objects, if they were objects at all, did not have many of the properties of macroscopic objects like heads of cabbage and bowling balls. Give me a good macroscopic object any day, fully separate and useful. Don't confuse me with these subatomic fantasies of interconnectedness. But what about thoughts?
Is my thought of a bowling ball an object? What about the mathematical description of the world where one has sets of objects forming new objects?

We can start anew from the dictum that the perceiver and the perceived arise together in the condition of observation. This is a stance that insists on mutuality (neither perceiver nor the perceived causes the other). A distinction has emerged and with it a world with an observer and an observed. The distinction is itself an eigenform.

IV. Shaping a World

We, identify the world in terms of how we shape it. We shape the world in response to how it changes us. We change the world and the world changes us. Objects arise as tokens of a behaviour that leads to seemingly unchanging forms. Forms are seen to be unchanging through their invariance under our attempts to change, to shape them.

For an observer there are two primary modes of perception -- *compresence* and *coalesence*. Compresence connotes the coexistence of separate entities together in one including space. Coalesence connotes the one space holding, in perception, the observer and the observed, inseparable in an unbroken wholeness. Coalesence is the constant condition of our awarness. Coalesence is the world taken in simplicity. Compresence is the world taken in apparent multiplicity.

This distinction of compresence and coalesence, drawn by Henri Bortoft [2], can act as a compass in traversing the domains of object and reference. *Eigenform is a first step towards a mathematical description of coalesence.* For in the world of eigenform the observer and the observed are one in a process that recursively gives rise to each.

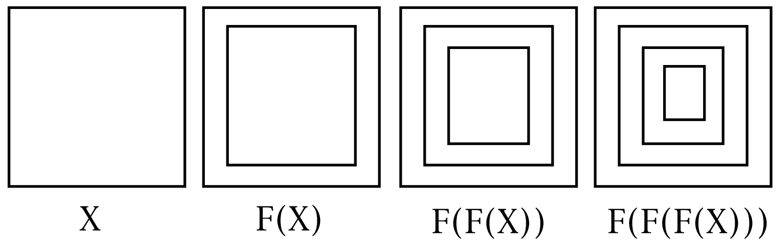
V. The Eigenform Model

We have seen how the concept of an object has evolved to make what we call objects (and the objective world) processes that are interdependent with the actions of observers. The notion of a fixed object has become a notion of a process that produces the apparent stability of the object. This process can be simplified in a model to become a recursive process where a rule or rules are applied time and time again. The resulting object of such a process is the *eigenform* of the process, and the process itself is the *eigenbehaviour.*

In this way we have a model for thinking about object as token for eigenbehaviour. This model examines the result of a simple recursive process carried to its limit.
For example, suppose that

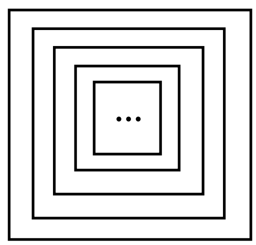
$$F(X) = \boxed{\quad X \quad}$$

That is, each step in the process encloses the results of the previous step within a box. Here is an illustration of the first few steps of the process applied to an empty box X:

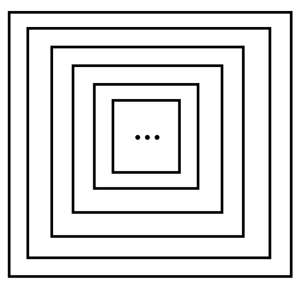


If we continue this process, then successive nests of boxes resemble one another, and in the limit of infinitely many boxes, we find that

X=F(F(F(...)))=



F(X) =



= X

the infinite nest of boxes is invariant under the addition of one more surrounding box. Hence this infinite nest of boxes is a fixed point for the recursion. In other words, if X denotes the infinite nest of boxes, then

$$X = F(X).$$

This equation is a description of a state of affairs. The form of an infinite nest of boxes is invariant under the operation of adding one more surrounding box. The infinite nest of boxes is one of the simplest eigenforms.

In the process of observation, we interact with ourselves and with the world to produce stabilities that become the objects of our perception. These objects, like the infinite nest of boxes, may go beyond the specific properties of the world in which we operate. They attain their stability through the limiting process that goes outside the immediate world of individual actions. We make an imaginative leap to complete such objects to become tokens for eigenbehaviours. It is impossible to make an infinite nest of boxes. We do not make it. We *imagine* it. And in imagining that infinite nest of boxes, we arrive at the eigenform.

The leap of imagination to the infinite eigenform is a model of the human ability to create signs and symbols. In the case of the eigenform X with $X = F(X)$, X can be regarded as the name of the process itself or as the name of the limit process. Note that if you are told that

$$X = F(X),$$

then substituting $F(X)$ for X, you can write

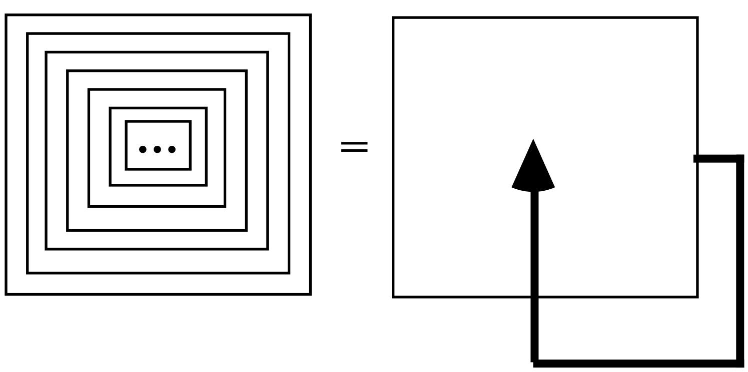
$$X = F(F(X)).$$

Substituting again and again, you have

$$X = F(F(F(X))) = F(F(F(F(X)))) = F(F(F(F(F(X))))) = \ldots$$

The process arises from the symbolic expression of its eigenform. In this view *the eigenform is an implicate order for the process that generates it.*

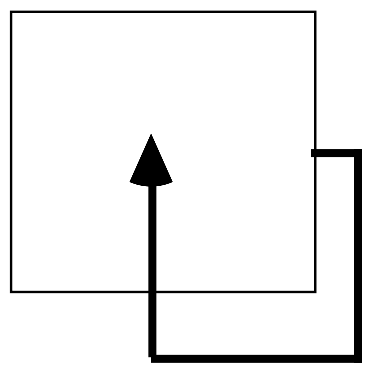
Sometimes one stylizes the structure by indicating where the eigenform X reenters its own indicational space by an arrow or other graphical device. See the picture below for the case of the nested boxes.



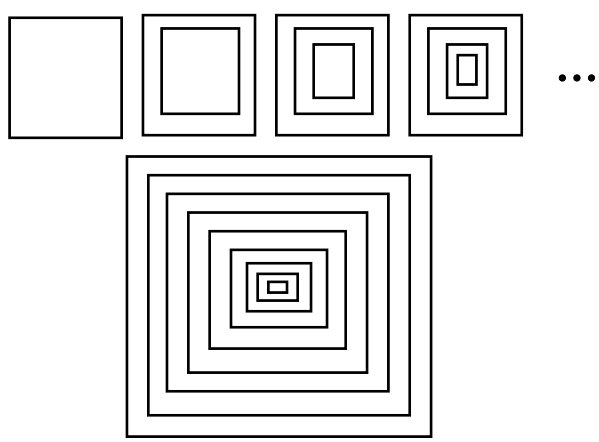
Does the infinite nest of boxes exist? Certainly it does not exist in this page or anywhere in the physical world with which we are familiar. The infinite nest of boxes exists in the imagination. It is a symbolic entity.

Eigenform is the imagined boundary in the reciprocal relationship of the object (the "It") and the process leading to the object (the process leading to "It"). In the diagram below we have indicated these relationships with respect to the eigenform of nested boxes. Note that the "It" is illustrated as a finite approximation (to the infinite limit) that is sufficient to allow an observer to infer/perceive the generating process that underlies it.

The It



The Process leading to It.



Just so, an object in the world (cognitive, physical, ideal,...) provides a conceptual center for the exploration of a skein of relationships related to its context and to the processes that generate it. An object can have varying degrees of reality just as does an eigenform. If we take the suggestion to heart that objects are tokens for eigenbehaviors, then an object in itself is an entity, participating in a network of interactions, taking on its apparent solidity and stability from these interactions.

An object is an amphibian between the symbolic and imaginary world of the mind and the complex world of personal experience. The object, when viewed as process, is a dialogue between these worlds. The object when seen as a sign for itself, or in and of itself, is imaginary.

Why are objects apparently solid? Of course you cannot walk through a brick wall even if you think about it differently. I do not mean apparent in the sense of thought alone. I mean apparent in the sense of appearance. The wall appears solid to me because of the actions that I can perform. The wall is quite transparent to a neutrino, and will not even be an eigenform for that neutrino. This example shows quite sharply how the nature of an object is entailed in the properties of its observer.

The eigenform model can be expressed in quite abstract and general terms. Suppose that we are given a recursion (not necessarily numerical) with the equation

$$X(t+1) = F(X(t)).$$

Here $X(t)$ denotes the condition of observation at time t. $X(t)$ could be as simple as a set of nested boxes, or as complex as the entire configuration of your body in relation to the known universe at time t. Then $F(X(t))$ denotes the result of applying the operations symbolized by F to the condition at time t. You could, for simplicity, assume that F is independent of time. Time independence of the recursion F will give us simple answers and we can later discuss what will happen if the actions depend upon the time. In the time independent case we can write

$$J = F(F(F(\ldots)))$$

the infinite concatenation of F upon itself. Then

$$F(J) = J$$

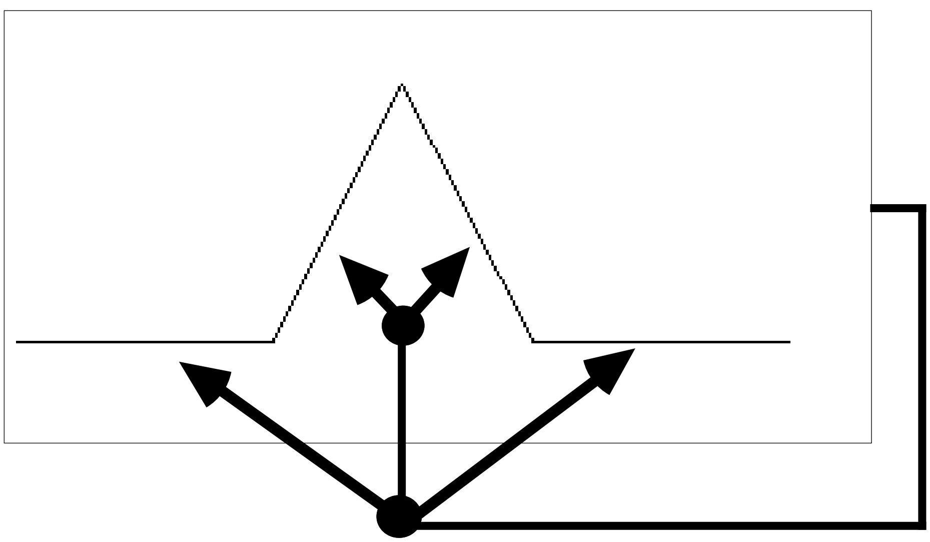
since adding one more F to the concatenation changes nothing. Thus J, the infinite concatenation of the operation upon itself leads to a fixed point for F. J is said to be the eigenform for the recursion

F. We see that every recursion has an eigenform. Every recursion has an (imaginary) fixed point.

We end this section with one more example. This is the eigenform of the Koch fractal [14]. In this case one can write symbolically the eigenform equation

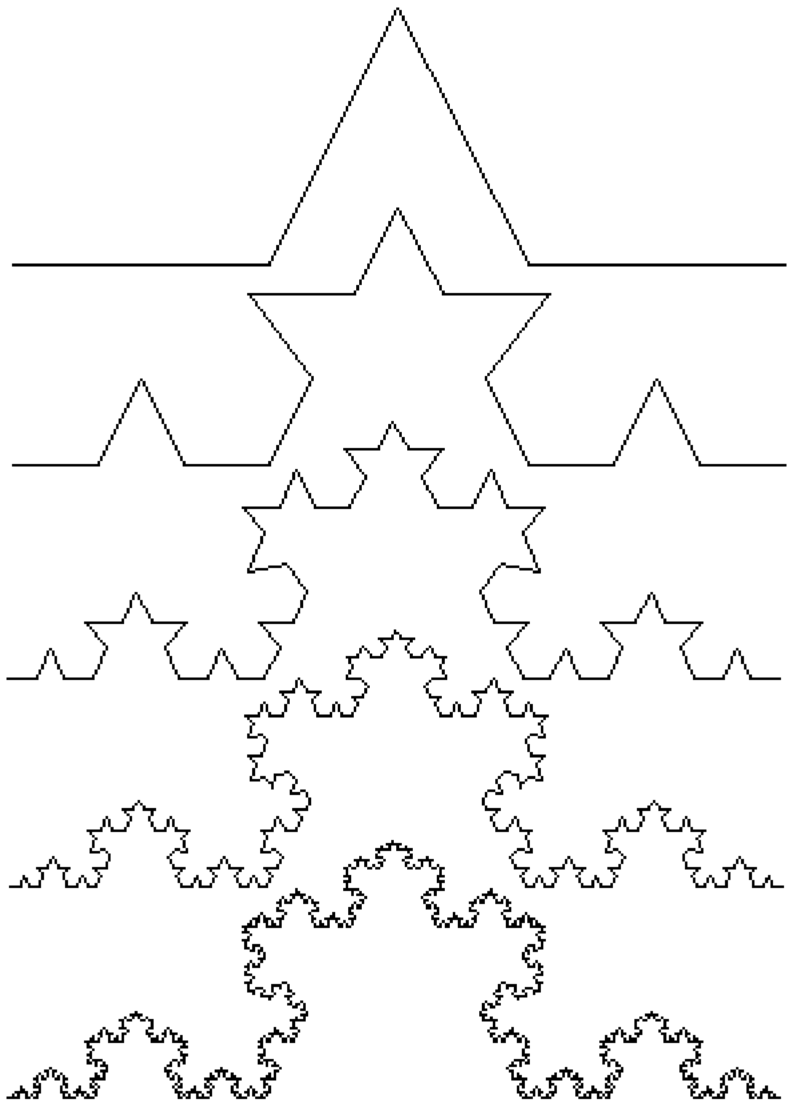
$$K = K \{ K \quad K \} K$$

to indicate that the Koch Fractal reenters its own indicational space four times (that is, it is made up of four copies of itself, each one-third the size of the original. The curly brackets in the center of this equation refer to the fact that the two middle copies within the fractal are inclined with respect to one another and with respect to the two outer copies. In the figure below we show the geometric configuration of the reentry.



$$K = K \{ K\ K \} K$$

In the geometric recursion, each line segment at a given stage is replaced by four line segments of one third its length, arranged according to the pattern of reentry as shown in the figure above. The recursion corresponding to the Koch eigenform is illustrated in the next figure. Here we see the sequence of approximations leading to the infinite self-reflecting eigenform that is known as the Koch snowflake fractal.



Five stages of recursion are shown. To the eye, the last stage vividly illustrates how the ideal fractal form contains four copies of itself, each one-third the size of the whole. The abstract schema

$$K = K \{ K\ K \} K$$

for this fractal can itself be iterated to produce a "skeleton" of the geometric recursion:

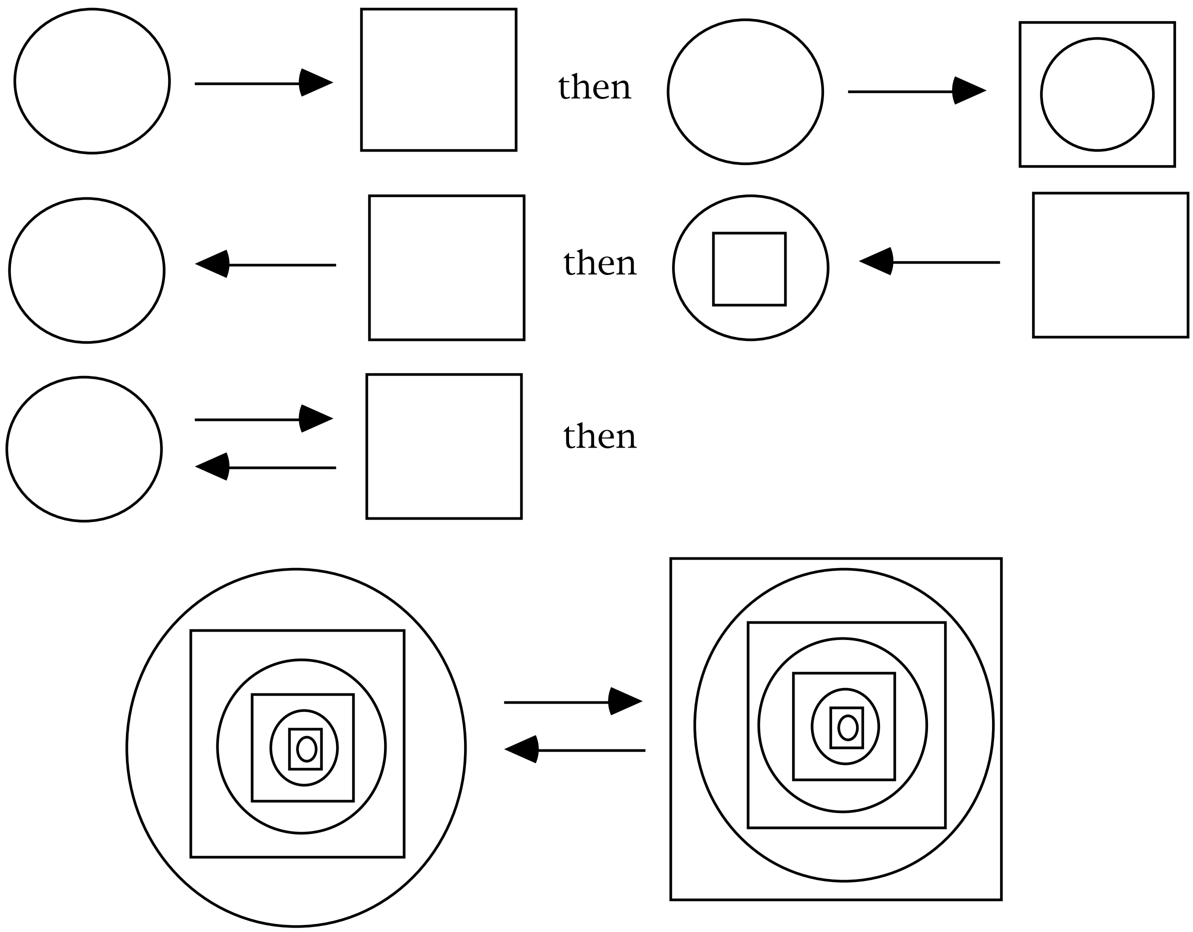
$$\begin{aligned} K &= K \{ K\ K \} K \\ &= K \{ K\ K \} K \ \{ \ K \{ K\ K \} K \ \ K \{ K\ K \} K \ \} K \{ K\ K \} K \\ &= \ldots \end{aligned}$$

We have only performed one line of this skeletal recursion. There are sixteen K's in this second expression just as there are sixteen line segments in the second stage of the geometric recursion. Comparison with this symbolic recursion shows how geometry aids the intuition. The interaction of eigenforms with the geometry of physical, mental, symbolic and spiritual landscapes is an entire subject that is in need of deep exploration.

It is usually thought that the miracle of recognition of an object arises in some simple way from the assumed existence of the object and the action of our perceiving systems. This is a fine tuning to the point where the action of the perceiver and the perception of the object are indistinguishable. Such tuning requires an intermixing of the perceiver and the perceived that goes beyond description. Yet in the mathematical levels, such as number or fractal pattern, part of the process is slowed down to the point where we can begin to apprehend the process. There is a stability in the comparison, in the correspondence that is a process happening at once in the present time. The closed loop of perception occurs in the eternity of present individual time. Each such process depends upon linked and ongoing eigenbehaviors and yet is seen as simple by the perceiving mind. The perceiving mind is itself an eigenform.

Mirror-Mirror

In the next figure we illustrate how an eigenform can arise from a process of mutual reflection. The figure shows a circle with a an arrow pointing to a rectangle and a rectangle with an arrow pointing toward a circle. For this example, we take the rule that an arrow between two entities (P -----> Q) means that the second entity will create an internal mage of the first entity (Q will make an image of P). If P -----> Q and Q ------> P, then each entity makes an image of the other. A recursion will ensue. Each of P and Q generates eigenforms in this mutuality.



In this example we can denote the initial forms by C (for circle) and B (for box). We have C -----> B and B -----> C. The rule of imaging is (symbolically):
If P -----> Q then P -----> QP.
If P < ------ Q, then PQ <------ Q.
We start with the mutual reference C < -----> B.
This condition of mutual mirroring can be described by two operators C and B:
C(P)= CP correesponds to C -----> P.
B(Q) = BQ corresponds to Q < ----- B.
We are soving the eigenform equations
C(Y) = X,
B(X) = Y.
We have the mirror-mirror solution
X = BCBCBCBC...,
Y = CBCBCBCB...,
just as in the Figure.

We are quite familiar with this form of mutual mirroring in the physical realm where one can have two facing mirrors, and in the realm of human relations where the complexity of exchange (mutual mirroring) between two individuals leads to the eigenform of their relationship.

Notice that the rule P -----> Q leading to P -----> QP can be intepreted various ways. We may think of Q "perceiving" P. We may think of P as the name of Q. In the latter case it is natural for P (the name) to be affixed to Q. In Section ? we shall formalize this second point of view by the pattern P -----> Q *shifts* to #P -----> QP. In this formalism we think of P as the name of Q, but #P is the "metaname" that results when we have gone through the process of fusing Q and her name. This is the process that we habitually apply to persons with names. Once I have met you and know your name, your appearance for me is directly linked with your name. The separate occurrence of your name in my mental space is at a different level than the direct appearance of "you with your name". That upleveled version of your name is the metaname. Application of this bit of the linguistics of naming to "I" gives insight into the nature of self-reference.

VI. A Conversation

Ranulph asked "Does every recursion have a fixed point?", hoping for a mathematician's answer. And I said first, "Well no, clearly not, after all it is common for processes to go into oscillation and so never come to rest." And then I said, "On the other hand, here is the

Theorem: Every recursion has a fixed point.

Proof. Let the recursion be given by an equation of the form

$$X' = F(X)$$

where X' denotes the next value of X and F encapsulates the function or rule that brings the recursion to its next step. Here F and X can be any descriptors of actor and actant that are relevant to the recursion being studied. Now form

$$J = F(F(F(F(...)))),$$

the infinite concatenation of F upon itself.
Then we see that

$$F(J) = F(F(F(F(F(...))))) = J.$$

Hence J is a fixed point for the recursion and we have proved that every recursion has a fixed point.//

And I went on to say that this theorem was in my view a startling magician's trick, throwing us into the certainty of an eigenform (fixed point) corresponding to any process and at the same time challenging us to understand the nature of that fixed point in some context that is actually relevant to the original ground of conversation. Ranulph agreed, and our emails settled back into the usual background hum.

VII. Church and Curry

In this section we point out that the construction of eigenforms can be accomplished without an idealized excursion to infinity. The method was invented by Alonzo Church and Haskell Curry [1] in the 1930's. This method is commonly called the "lambda calculus". The key to lambda calculus is the construction of a self-reflexive language, a language that can refer and operate upon itself. In this way eigenforms can be woven into the context of languages that are their own metalanguages, hence into the context of natural language and observing systems.

In the Church-Curry language (the lambda calculus), there are two basic rules:

1. **Naming.** *If you have an expression in the symbols in lambda calculus then there is always a single word in the language that encodes this expression.* The application of this word has the same effect as the application of the expression itself.

2. **Reflexivity.** *Given any two words A and B in the lambda calculus, there is permission to form their concatenation AB, with the interpretation that A operates upon or qualifies B.* In this way, every word in the lambda calculus is both an operator and an operand. The calculus is inherently self-reflexive.

Here is an example. Let GA denote the process that creates two copies of A and puts them in a box.

$$GA = \boxed{AA}$$

In lambda calculus we are allowed to apply G to itself. The result is two copies of G next to one another, inside the box.

$$GG = \boxed{GG}$$

This equation about GG exhibits GG directly as a solution to the eigenform equation

$$X = \boxed{X}$$

thus producing the eigenform without an infinite limiting process.

More generally, we wish to find the eigenform for a process F. We want to find a J so that F(J) = J. We create an operator G with the property that

$$GX = F(XX)$$

for any X. When G operates on X, G makes a duplicate of X and allows X to act on its duplicate. Now comes the kicker.
Let G act on herself and look!

$$GG = F(GG)$$

So GG is a fixed point for F.

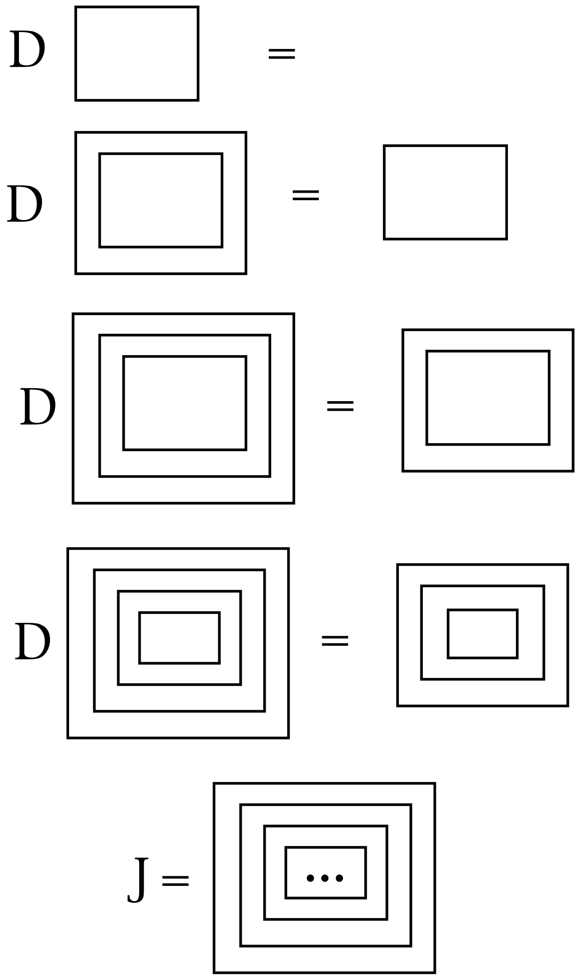
We have solved the eigenform problem without the excursion to infinity. If you reflect on this magic trick of Church and Curry you will see that it has come directly from the postulates of **Naming** and **Reflexivity** that we have discussed above. These notions, *that there should be a name for everything,* and that *words can be applied to* *the description and production of other words*, allow the language to refer to itself and to produce itself from itself. The Church-Curry construction was devised for mathematical logic, but it is fundamental to the logic of logic, the linguistics of linguistics and the cybernetics of cybernetics.

I like to call the construction of the intermediate operator G, the "gremlin" (See [10].) Gremlins seem innocent. They just duplicate entities that they meet, and set up an operation of the duplicate on the duplicand. But when you let a gremlin meet a gremlin then strange things can happen. It is a bit like the story of the sorcerer's apprentice. A recursion may happen whether you like it or not.

An eigenform must be placed in a context in order for it to have human meaning. The struggle on the mathematical side is to control recursions, bending them to desired ends. The struggle on the human side is to cognise a world sensibly and communicate well and effectively with others. For each of us, there is a continual manufacture of eigenforms (tokens for eigenbehaviour). Such tokens will not pass as the currency of communication unless we achieve mutuality as well. Mutuality itself is a higher eigenform. As with all eigenforms, the abstract version exists. Realization happens in the course of time.

VIII. Differentiation Creates Number

Consider an operator D that *removes* a box from around X.



$$DJ = J$$

Our familiar infinite nest of boxes is an eigenform for the "differentiation" operator D.

Calculus

The exponential function is invariant under differentiation. Thus it is an eigenform for the operator D=d/dt:

D(exp(t)) = exp(t) where D=d/dt.

In fact,

$$\exp(t) = 1 + t/1! + t^2/2! + t^3/3! + \ldots$$

where

$D1 = 0,$
$Dt^{(n+1)}/(n+1)! = t^n/n!$

from which it follows that

$D(\exp(t)) = \exp(t).$

If we think of the exponential function as a nest of boxes, each of which corresponds to one of the terms $t^n/n!$, then we see that the invariance of the nest of boxes J under its differentiation operator has exactly the form of the invariance of $\exp(t)$ under differentiation in the calculus.

Constructing Numbers

Consider the construction of numbers via sets. Each number is the collection of all the previously created numbers. We begin with nothing, collecting it into the empty set. Then 1 is the collection whose member is the empty set. The process continues ad infinitum.

0={ }
1={ { } }
2={ { }, { { } } }

It is a recursive process where

0={ }

and

n +1 = {0,1,2,3,...,n} = {Dn, n}

where *DS is the list obtained from a set S by removing its outer bracket.*

Note that D{ } is nothing, D{ { } } = { },
D{ { },{ { } } } = { },{ { } } and so on.

For the limit singleton W={W} we have DW = W. We now have the basic equation

N + 1 = {DN, N}.

In this form, the counting process resists the production of fixed points. For example, if we let

$I = \{0,1,2,3,...\}$

be the first ordered countable infinity of integers, then

$I + 1 = \{DI, I\} = \{0,1,2,3,..., I\}$

is a new set distinct from I. $I + 2$ is distinct from $I + 1$. The counting process continues infinitely.

IX. The Object of Set Theory

Lets look at objects from the point of view of a set theoretician. If A and B are objects, then we can form a *new object* C = {A,B}, the set consisting of A and B. This seems harmless enough. After all, if Chicago and New York are objects, then the set of large coastal cities in the United States should also be an object, albeit of a different type. We give up something with these mathematical objects. We do not assume that they have specific spatial locations. After all, what is the spatial location of the set {Chicago, New York}?

Take New York. This is a good big object to talk about. It is a place. It has a location. It has contents, all the people in it, all the goods and people and ideas and music running through it.
And we will leave all that and just take the set theoretic point of view and look at the singleton set {New York}. Now {New York} is not New York. Not by a long shot! New York is a hustling bustling metropolis on the East Coast of the United States of America. New York has millions of inhabitants and buildings, and New York is constantly changing. On the other hand, the singleton {New York} has exactly one member. It never changes. It is always the set whose member is New York. On top of this, once we have admitted the singleton object {New York} into existence, we are compelled to allow the singleton of its singleton to come on the stage with its only member the singleton of New York: {{New York}}. There is a infinity of singleton objects derived from New York waiting in the wings:

{New York}
{{New York}}

{{{New York}}}
{{{{New York}}}
...
{{{{{{{{{{{{{{{{{New York}}}}}}}}}}}}}}}}}
...
...
{{{{{{{{{{{{{{{{{{{{{{{...}}}}}}}}}}}}}}}}}}}}}}} = N

The limiting New York singleton W has New York infinitely down in the nest of parentheses. New York has disappeared and all that is left of the Cheshire Cat is its grin. All that is left in the limit W is the fact that W is invariant under the act of forming the singleton. We have that W is its own singleton!

N = {N}

Adding one more level of parentheses makes no difference.
N is at a level where the level and the metalevel are one.
N is both object and subject of the set theoretic discourse.
And if you think that N has nothing to do with New York you are wrong! W is the very identity of New York. N is the ultimate singleton associated with New York. W is the essence of New York. And at the same time N is entirely content free and has nothing to do with New York. N is just an infinite nest of brackets. An uninterpreted bit of self-reference in the void. Will you have it both ways? You could locate N anywhere. Why not New York?

X. Singletons and Eigenforms

The example of New York illustrates the extreme eigenform associated with any object in the set theoretic universe. We can iterate the operation of framing X to form the singleton {X} ad infinitum and lose X in the infinite depths of the recursion. We lose X and regain the ubiquitous and self-referential N = {N}. This could bring one to be suspicious of the concept of singleton set. After all, why should New York or any proper object in the world be surrounded by an infinite halo of singletons? When I eat an apple, must I devour {apple}, {{apple}}, {{{apple}}}, ... as well? Quine and others have suggested that we take a different approach to framing so that singletons do not appear. One way to achieve this is to legislate that {S} = S for any S that is non-empty.

Think about this proposal. We would have {{}} = {} and there would be no way to produce a set with one element! One could say that there are special objects in the theory, lets call them a,b,c, ... such that each object is its own singleton:

{a}=a, {b} = b, {c} = c, ...

Then, at least the singletons for the "real" objects collapse back to them. This approach raises many questions. What are the special objects. Certainly no mathematician would want the empty set to be among them, since we wish to discriminate between the empty set and the set whose member is the empty set. Searching for these special objects is something like searching for elementary particles. Where are they? Could it be that I am a special object? Lets see.
Is it the case that I = {I}. Why yes indeed? I can frame (think about) myself and I am still myself! In fact if we interpret the emergence of a frame as an act of reflection (thought), then the special objects appear as elements of I-ness, as signs for themselves in the sense of Charles Sanders Peirce [10].

But set theory goes its own way, and would weave these special objects together into hierarchies that embody singletons once again. Take two specials a and b. Form their union {a,b}. Is this special? Why not? Why not allow that if any element of a set S is special, then S is also special? This still allows room for classical mathematics. We can always form sets like {{},{{}}} that are not special. Note that a = {a} does not imply that a is equal to an infinite nest of parentheses. The infinite nest W = {{{{{{{...}}}}}}} is but one of many special objects that are their own singletons.

By including special objects into set theory and these rules for their composition, we have created a model for a set theoretic world that contains a parable (a parallel parable!) of the mental and the physical. The purely mental world is the class of sets generated from the empty set. The purely physical world is the class of sets generated from the special objects. The interface of mental and physical occurs as they touch in the limit of nested parentheses.
W = {{{{{{{...}}}}}}} is an amphibian living in both worlds. W is the eigenform that crosses the boundary from the mental to the physical.

XI. The Form of Names and Godel's Theorem

Here we consider naming and self-reference and return to von Foerster's definition of "I". The concept of this section is essentially related to the lambda calculus where names can act on names and their referents. We discuss how self-reference occurs in language through an indicative shift welding the name of a person to his/her (physical) presence and shifting the indication of the name to a metaname. More could be said at this point, as the indicative shift is a linguistic entry into the world of Godelian sentences and the incompleteness of formal systems. We emphasize the natural occurrence of eigenforms in the world of our linguistic experience and how this occurrence is intimately connected to our structure as observing systems.

The simplicity of a thought, the apparent clarity of distinction is mirrored in the sort of eigenforms that come from the Church-Curry realm. Consider a linguistic example: Each person has a name (at least one). In the course of time we are introduced to people and come to know their names. We know that name not as an item to look up about the person (and this applies to certain objects as well) but as a direct property of the person. That is, if I meet Heinz he appears to me as Heinz, not as this person with certain characteristics, whose name I can find in my social database if I care to do so. It is like this only when we are first introduced. At the point of introduction there is this person and there is his name separate from him. Once learned, *the name is shifted* and occurs in space right along with the person. Heinz and his name are in the same cognitive space which is also in the same place as the apparent physical space. We can observe this shifting process in the course of learning a name. We can also observe how physical and cognitive spaces are superimposed. The many classical optical illusions illustrate these matters vividly.

Now we have Heinz with his name inseparable from his presence, and this is true even if he is not physically present, for the shift has occurred and will not be undone. But we also have his name Heinz separate from him, and able to be pinned upon another. And we have his name not quite separate from him, but rather this Heinz is the name of the name we have attached to him! This is Heinz's metaname. How do we distinguish among all these different names for Heinz? We use the same symbols for them, yet they are different. Lets choose a way to indicate the differences.
We start with the reference.

Heinz -----> Cybernetic Magician

(The arrow will indicate that the entity on the left is the name of the entity on the right.)

We get to know him and shift the reference.

#Heinz -----> Cybernetic Magician Heinz

Now the name is in the cognitive space of Heinz, and the metaname #Heinz refers to that conjunction. We shall call this **the indicative shift.**

name -----> object
#name-----> object name

The indicative shift occurs, constantly weaving the apparent external reality with the linguistic reality.

Self-reference occurs when one names the metanaming operator.

At the very first we have void referring to void.

----->

Shifting, we have

----->

Thus, # refers at first to the singular place where there is an absence of naming, a void in the realm of distinctions.

Then the shift occurs again. We have the reference of the meta-name of the meta-naming operator to itself (as the operator enters a space formerly void).

----->

Next we have

###-----> ###

a self-reference at the third departure from the void.

After this, the shifting produces a exponential increase in strings in the operator but no more self-reference.

---->######
---->##########

---->###############

...

Suppose that the meta naming operator has any other name, say M. Then we have

M -----> #

which shifts to a self-reference at the second articulated level of meta naming.

#M -----> #M

These are the eigenforms of self reference in the realm of names. To justify this dictum, think about what is an "I".

An "I" makes names and also shifts names to the entities so named.

Thus the "I" of our individual experience is our metanaming operator and so acts to blend whatever name we give it , with "itself". In so doing it must refer to itself

M-----># to #M ----->#M.

We can state the identity

I = #M,

for i*t is exactly at the coallesence of the meta-naming operator and ithe name of the metanaming operator that self-reference occurs.*
I = #M is the eigenform of my linguistic idenity.

This is the linguistic structure by which the on-going text of my speaking comes to refer it itself and appear to have consciousness. T*his account of consciousness assumes a substrate from which naming and the shifting of names arises.* This substrate might be explained by the neurological or biological basis of the organism, but we do not explore these avenues here.

Von Foerster gave a cybernetic definition of "I"[6]:

"I am the observed link between myself and observing myself."

We encourage the reader to compare his definition with our linguistic characteriztion.

Godelian Self-Reference

Self-reference at this level, the action of a domain upon itself, leading to cognition, is the beginning of the realm of eigenforms.

The indicative shift is the key mechanism behind Godel's Theorem on the incompleteness of formal systems. Suppose that

g -----> F#

then

#g -----> F#g.

Thus F#g speaks about its own name. This is the pattern of the Godelian self reference. In Godel's context the names g are code numbers for statements in a formal system **S**. This system **S** is designed to talk about numbers. Thus **Fu** can be any mathematical statement about the number **u**. In particular we can take **F = ~B** where **Bu** means that "the statement whose code number is **u** has a proof in **S**". Then **~Bu** means that "the statement whose code number is **u** does not have a proof in **S**." So if we form the shift

g -----> ~B#

#g -----> ~B#g

we arrive at the statement **~B#g** that means "there is no proof in **S** of the decoding of **#g**". But **~B#g** is the decoding of **#g**. So finally we have **~B#g** asserts its own unprovability in **S**! This is the core of Godel's Incompleteness Theorem.

It is amazing that the natural acts of *coding* , *mirroring* and *naming* that occur in our everyday language are the source of the deep incompleteness results for the mathematics of formal systems as well as the source of our idea of self. That idea is central eigenform of this paper.

XII. Cantor's Diagonal Argument and Russell's Paradox

Let AB mean that *B is a member of A.*

Cantor's Theorem. Let S be any set (S can be finite or infinite). Let P(S) be the set of subsets of S. Then P(S) is bigger than S in the sense that for any mapping F: S -----> P(S) there will be subsets C of S (hence elements of F(S)) that are not of the form F(a) for any a in S. In short ,the power set P(S) of any set S is larger than S.

Proof. Suppose that you were given a way to associate to each element x of a set S a subset F(x) of S. Then we can ask whether x is a member of F(x). Either it is or it isn't. So lets form the set of all x such that x is not a member of F(x). Call this new set C. We have the defining equation for C :

$$Cx = \sim F(x)x.$$

Is C =F(a) for some a in S?
If C=F(a) then for all x we have
F(a)x = ~F(x)x.
Take x =a. Then
F(a)a = ~F(a)a.
This says that a is a member of F(a) if and only if a is not a member of F(a). This shows that indeed C cannot be of the form F(a), and we have proved Cantor's Theorem that the set of subsets of a set is always larger than the set itself. //

Note the problem that the assumption that C = F(a) gave us.
If C = F(a), then F(a)a = ~F(a)a. We would have a fixed point for negation. The mark of a contradiction in a classical argument is the appearance of a fixed point for the negation operator. When a eigenform for "not" appears on the scene, we run for the hills!

Of course the point is that in classical two-valued logic there is no fixed point for negation. If we had enlarged the truth set to

$$\{T, F, I\}$$

where ~I=I is an eigenform for negation, then F(a)a would have value I. What does this mean? It means that the index a of the set F(a) corresponding would have an oscillating membership value. The element a would be like Groucho Marx who declared that he would not join any club that would have him as a member. We would be propelled into sets that vary in time.

Note also how close Cantor's Theorem is Russell's famous paradox. Russell devised the set R defined by the equation

$$Rx = \sim xx.$$

An element x is a member of the Russell set if and only if x is not a member of itself.

To see the contradiction, substitute R for x and get

$$RR = \sim RR.$$

This appearance of an eigenform for negation tells us that we either must concede temporality to Russell's construction **R**, or else banish it from the world of sets. The standard solution is banishment, since classical mathematics wants a timeless world of eternal forms.

Lets go back to Cantor's Theorem. Let $S = \{1,2,3,4,...\}$ be the natural numbers, a countably infinite set. Then Cantor's proof shows that the set of subsets of S, P(S), is uncountable. How big is P(S)?
Is there a set X such that $S < X < P(S)$? This question is *Cantor's continuum problem.* It is called the continuum problem because $P\{1,2,3,4,...\}$ can be mapped to our models of a continuous line (think of points on the line as infinte decimals). The continuum is an eigenform that is part of our way of being and it is, once examined, an idealization of experience that only has existence as an idea and yet is part of our life and action.

It was shown by Paul Cohen that Cantor's continuum problem is independent of the present axioms for set theory. There are models where there is an intermediate set X and other models where there is no intermediate set X. The difficulty lies in the fact that Cantor's eigenform argument is really the only way we have to show that one infinte set is bigger than another. Our ability to handle infinity with finite language is more restricted than we had imagined. This indicates a domain for deeper investigation of the eigenform concept, in the world of comparision of nfinities.

XIII. Knot Sets and Topological Eigenforms

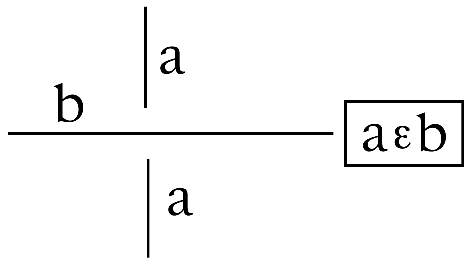
We shall use knot and link diagrams to represent sets. More about this point of view can be found in the author's paper "Knot Logic" [9]. In this notation the eigenset Ω satisfying the equation

$$\Omega = \{\Omega\}$$

is a topological curl. If you travel along the curl you can start as a member and find that after a while you have become the container. Further travel takes you back to being a member in an infinite round. In the topological realm Ω does not have any associated paradox. This section is intended as an introduction to the idea of *topological eigenforms*, a subject that we shall develop more fully elsewhere.

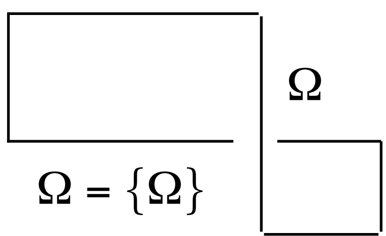
Set theory is about an asymmetric relation called *membership.*

We write a ε S to say that a is a member of the set S.   In this section we shall diagram the membership relation as follows:



This is *knot-set notation.*
In this notation, if b goes once under a, we write a={b}. If b goes twice under a, we write a={b,b}. This means that the "sets" are multi-sets, allowing more than one appearance of a member. For a deeper analysis of the knot-set structure see [KL].

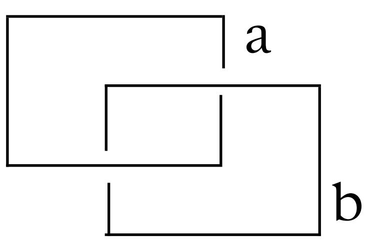
This knot-set notation allows us to have sets that are members of themselves,



$\Omega = \{\Omega\}$

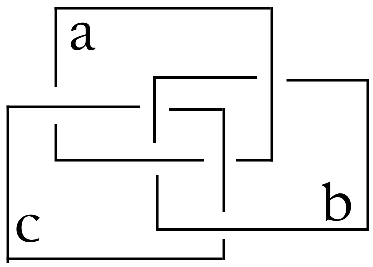
$\Omega \ \varepsilon \ \Omega$

and sets can be members of each other.



a={b}
b={a}

Here a mutual relationship of a and b is diagrammed as topological linking.



$$a = \{b,b\}$$
$$b = \{c,c\}$$
$$c = \{a,a\}$$

Here are the *Borromean Rings.* The Rings have the property that if you remove any one of them, then the other two are topologically unlinked. They form a topological tripartite relation. Their knot-set is described by the three equations in the diagram.
Thus we see that this representative knot-set is a "scissors-paper-stone" pattern. Each component of the Rings lies over one other component, in a cyclic pattern.

XIV. In Zermelo's Bar

The section is a multi-logue about the attempts to solve the equation of the obsever in relation to his/her observation. We first encounter Mr. D, who has solved his own equation in such a way that he has no head and instead has a great open space of possibility where his head was supposed to be. This requires a drink to ingest and we go to Zermelo's Bar, where we find two mathematicians arguing over the solution to an equation whose solution is the Golden Ratio, a proportion well known to the Greeks. The mathematicians are a little hard to follow, but their discussion turns on all the essential issues of recursion, reality and infinity that we will need for this adventure. Then Dr. Von F appears in the bar (we think you can guess who this is) and explains the nature of eigenforms. He is followed by a character named Charlie and Dr. CC, a linguist and logician, then by Dr. HM, a biologist. Later there appears a physicist, Dr. JB and finally Dr. R himself, the source of the self-referential paradox. We hope that you will join in on this discussion yourself.

Infinite Recursion and Its Relatives

Our problem is to solve the equation

$$O(A) = A$$

for A in terms of O.

For example, suppose that the observer O is Mr. D, a man who insists that he has no head. We interview him. Well Mr. D, why do you say that you have no head? Mr. D. replies. Oh it is so simple, you will see at once what I mean. In fact, consider what you yourself see. Look directly around. Do you see your head? No. You see and feel a great open space of perception where your head is supposed to be, and a flow of thoughts and feelings. But no head! The body comes in. Shoulders, arms, legs, shoes and the world. But no head. Instead of a head there is a great teeming void of perception. Once I realized this, I knew that the relationship of a self to reality was indeed deep and mysterious.

As we can see, Mr. D has discovered that what is constant for his visual observer is a body without a head. He has solved the problem of finding himself as a solution of the equation of himself in terms of himself. Perhaps we need a drink.

We walk into Zermelo's Bar and two mathematicians appear on the scene. One says to other: How do you solve this equation? I want a postive real solution.

$$1 + 1/A = A.$$

The second one says:
Nothing to it, we multiply both sides by the unknown A and rewrite as

$$A + 1 = A^2.$$

Then, solving the quadratic equation, we find that

$$A = (1 + \sqrt{5})/2.$$

The first mathematician says: Nice tricks you have there, but I prefer infinite reentry of the equation into itself. Look here:

If $A = 1 + 1/A$, then

$$\begin{gathered} A = \\ 1 + 1/A = \\ 1+1/(1+1/A) = \\ 1+1/(1+1/(1+1/A)) = \\ 1+1/(1+1/(1+1/(1+1/A))) \end{gathered}$$

and I will take this reentry process to infinity and obtain the form

$$A = 1+ 1/(1+1/(1+1/(1+1/(1+ 1/(1+1/(1+\ldots)))))).$$

The second mathematician then says: Well I like your method. We can combine our answers and write a beautiful formula!

$$(1+\sqrt{5})/2 \ =$$

$$1+ 1/(1+1/(1+1/(1+1/(1+ 1/(1+1/(1+\ldots))))))$$

Why do you like this formula? says the second guy. Well, sez the first guy, the left hand side is a definite irrational number and it is easy to see by squaring it that it satisfies the equation $A^2 = A + 1$ as we wanted it. But irrational numbers have a curiously tenuous existence unless you know a way to calculate approximations for them. On the other hand, your right hand side can be regarded as the limit of the fractions

$1 = 1/1$
$1+1/1 = 2/1 = 2$
$1+1/(1+1/1) = 3/2$
$1+1/(1+1/(1+1/1)) = 5/3$
$1+1/(1+1/(1+1/(1+1/1))) = 8/5$
$1+1/(1+1/(1+1/(1+1/(1+1/1)))) = 13/8$
$1+1/(1+1/(1+1/(1+1/(1+1/(1+1/1))))) = 21/13$

with the first few terms of this limit being

$$(1+\sqrt{5})/2 = 1.618\ldots$$

On top of this your infinite formula actually does reenter itself as an infinite expression it really is of the form

$$A = 1 + 1/A.$$

The first guy comes back with: Well it sounds to me like you really believe in the "actual" infinity of the terms on the right-hand side. I also like to imagine that they are all there existing together in space with no time.

Right ! says the second guy. We know that this is an idealization, but it lets us actually reason to correct answers and to put them in an aesthetically pleasing form.

The bartender is listening to all of this, and he leans over and says: You guys have to meet a couple of others on this score. There is Dr. Von F and Dr. CC. They both have some ideas very similar to yours. Hey, here is Dr. Von F now. Dr. Von F, could you tell these fellows about your eigenforms?

Jah! Of course! It is all very simple. We just combine this notion of recursion with the most general possible situation. Suppose we have any observer O and we wish to find a fixed point for her. Well then we just let the observer act without limit as in

$$A = O(O(O(O(O(O(O(O(\ldots)))))))).$$

After infinity, one more application of O does not change the result and we have

$$O(A) = A.$$

This is very simple, no? And it shows how we make objects. These objects are the tokens of our repeated behaviours in shaping a form from nothing but our own operations. As I have said before, the human identity is precisely the fixed point of such a recursion. "I am the link between myself and observing myself." [2]

The first mathematician makes a comment: What you are doing is a precise generalization of my infinite continued fraction! If I had defined

$$O(A) = 1 + 1/A$$

then we would have

$$O(O(O(\ldots))) = 1+ 1/(1+1/(1+1/(1+\ldots))).$$

But I am puzzled by your approach, for it would seem that you are willing that your solution A will have no relation with how the process starts, and also it may not related to the original domain in which it was constructed!
For example, in my mathematics, I could consider the operator

$$O(A) = -1/A$$

and this operator does not have a fixed point in the real numbers, but if we take $A=i$ where $i^2 = -1$ (the simplest imaginary number), then $O(i)=i$. Are you suggesting that

$$i = -1/-1/-1/... \quad ?$$

Dr. Von F replies: Jah, Jah! This is very important! The fixed point can be a construction that breaks ground into an entirely new domain! Actually, I am mainly interested in those fixed points that do break new ground. We are looking for the places where new structures emerge. In your mathematics you have illustrated this in two ways. In the first recursion, the values converge to an irrational number (the golden ratio). All the finite approximations are rational fractions (ratios of Fibonacci numbers) but in the limit of the infinite eigenform, you arrive at this beautiful new irrational number! And in your second example all the finite approximations oscillate like a buzzer, or a paradox, between positive unity and negative unity, but the eigenform is a true representative of the imaginary square root on minus one! And don't forget that this "imaginary" quantity is fundamental to both logic and physics. The fully general eigenforms are fundamental to the ontology of the world.

Suddenly the door to Zermelo's Bar opens and in walks a character that everyone calls "Charlie." Charlie! say the barkeep, where have you been? We have a good discussion on signs going here. You have to hear this stuff. Charlie says, Well I heard just about everything Dr. Von F said as I admit here to a bit of eavesdropping on the other side of the door! These eigenforms of Von F are quite familiar to me as I have thought continuously along these lines for many years. You see, any sign once you look at it in the cotext of its reference and the continous expansion of its interpretant becomes a growing complex of signs referring to other signs, growing until the references close on themselves and, as Dr. Von F correctly describes, these closures are the eigenforms, the tokens for appaently stable behaviours. As the complex of signs grows, the complex itself is a sign and as the closures occur that sign becomes a sign for itself. We humans are in our very nature such signs for ourselves.

Dr. Von F says: Well I always say that I am the link between myself and observing myself. I am a sign for myself!

At this point Dr. CC chimes in: But Dr. Von F and Charlie, this excursion to recursion and infinity seems quite excessive! It is all right for mathematicians to imagine such a thing, but we humans exist in language and the finiteness of expressions. Surely you do not suggest that this profligate composition of the operator and expansion of sign complexes actually happens!

Well, Dr. CC, says Von F, I am really a physicist and well aware of the speed of physical process in relation to the very slow pace of our verbal thought. Surely you have stood between two facing mirrors and seen the near-instantaneous tunnel of reflections created by light bouncing back and forth between the mirrors. Yes, I am seriously suggesting that the self-composition of the observer is carried to high orders. These orders are sufficiently large and accomplished with such a high speed that they appear infinite in the eyes of the observer. Now you may detect the beginning of a paradoxical flight here. The very observer who is too slow to detect the difference between a large number and infinity is yet so quick and subtle that he/she can produce this flight to infinity. But I beg your pardon, this is still a matter of the interaction of slow thought and fast action. Wave your arm back and forth rapidly in front of your eyes. For all practical purposes the arm appears to be in two places at the same time! You do not deny that it is "you" that moves the arm, and it is "you" that perceives it.
I simply go further and suggest that every perception is based on such an illusion of permanancy, based on the self composition of your self. You do it all and you are surprised at the result. You do it all, but you can not perceive all that you do!

Charlie adds: I agree but do not have to rest on physics. Our shortsighted view of our own nature arises from the difficulty in reckoning that our true nature is as signs for ourselves. It is only at the limit of eigenbehaviours that such signs appear simple. We partake of the complexity of the universe.

Dr CC replies: Ah Charlie and Dr. Von F, I have been working in the linguistic and logical realm and you will see that our points of view are mutually supporting. For I imagine the structure of the observer as a big network of communicating entities. These entities have so much interrelation among themselves that their identities begin to merge into one identity and that is the apparent identity of the self.

Charlie interrupts with: Yes! That is the essence of continuity.

Dr. CC continues. I agree! The infinity in my view is not with any one of them, but with the aggregate of them that has become so large as to begin to merge into a continuity.

But let me explain: If A and B are entities in my "community of the self", then they can interact with each other and with themselves. These processes of interaction produce new entites who exist at the same level as the original entities. Can you imagine this? Of course you can, you are such an entity. For example, I suggest to you that you are the self that thinks kindly of others, that you satisfy the equation $SX = KX$ where S is "you" and KX is the being "thinks kindly of X". *Then that entity S exists.* In the world of language, *every definable entity exists*. The consequence is that$S$ might even think kindly of herself as in $SS = KS$. That S can think kindly of herself is, in this linguistic world, dependent on the condition that the kindly thinking observer is an observer at the same level as any other observer. Now there are many such entities. Watch this magic trick. Let

$$GX = O(XX).$$

The entitity G is the observer who observes an entity observing herself. What happens when G observes herself? Then G observes herself observing herself and we have a fixed point, an eigenform!

$$GG = O(GG).$$

I have constructed the eigenform without the infinite composition of the observer upon herself. Of course once this self-reflexive construction comes into the being of language then it runs automatically to the level of practical infinity and produces your recursion.

$$GG = O(GG) = O(O(GG)) = O(O(O(GG))) = \ldots$$

I believe my linguistic construction provides the context for your observer's self interaction. The true infinity in my world is a distributed infinity of beings each coming into being as a name for a process of observation. This contines without end and is the basis of the coincidence of the language and the metalanguge in this world.

At this point Dr. HM, a biologist, walks into the room. He remarks: I see that you have been discussing the stability of perceptions from physical and linguistic principles. Let me tell you how I see these matters in my domain. The beings you talk about are biological, not just logical. They exist in the evolutionary flow of coordinations of coordinations that give rise to the mutual patternings that you call "language" and "thought". It is not at all surprising that each such being, coordinated with the others in the deep flow of its history in biological time will appear layered like an onion with the actions of each on each. The long time history of mutual interaction and coordination will generate the appearance of the eigenforms. But there is no "disembodied observer" who generates these forms from some abstract place. In biology there is no problem of mind (abstract observer) and body. They are one. Mind and observer both refer to the conversational domain that arises in the construction of the coordination of coordinations that is language. The disembodied observer is a fantasy that is convenient for the mathematician or the physicist. In the biological realm all forms are generated through time in an organic way.

And finally, Dr. JB enters the room, a very theoretical physicist. He says: Ah it is not surprising, but you all have the business of objects and eigenforms quite wrong. Let me start with the views of the biologist Dr. HM. You see, there is no time. None. Time is an illusion. Of course in order to tell you about this insight I shall have to use words that appear to describe states in time. That is my fate to be so projected into language. You must forgive me.
Each moment of being is eternal, beyond time. I prefer to call such moments "time capusules." Each moment contains that possibility that it can be interpreted in terms of a "history", a story of events leading up to the "present moment" that constitutes the time capsule as a whole. But this history is a pattern in eternity. That the history can be told with some coherence and that we manage to tell the story of "past events" leads us to believe that these past events "actually happened". But in fact what has happened is happening now and only now in the eternity of the time capsule whose richness dervives from the superposition of its quantum states.

At this point the bartender chimes in: I'll drink to that. Time is a grand illusion and a wee scotch from my bar will convince ye o' that in less time than it takes to wink an eye!

All well and good, says Dr. R, who just walked into the bar, but as I was telling my friend Frege, if there is one thing that will give us trouble it is this notion of eternity and the non-existence of time. For as I told Gottlob just the other day, you have only to imagine the timeless reality of the set of all sets that are not members of themselves and you will have to leave logic behind! I gave up long ago my travails on this issue with Professor Whitehead. We tried to make logic go first and it was a disaster. Now I let logic run along behind and there is no problem at all. As far as fixed points are concerned my favorite is Omega, the set whose only member is Omega herself. You see that the act of set formation is nothing but an act of reflection. Omega finds herself in reflecting on herself.

Dr. CC retorts: Well, Russell, I hardly expected you to capitulate your position on logic. Your Type is hardly likely to just slip away. I prefer to make a specimen of your famous set in the following way. I let **AB** mean that "B is a member of A". Then I define your set of all sets that are not members of themselves" by the equation

$$Rx = \sim xx.$$

Then we can pin the specimen to the board by substituting **R** for **x** as in

$$RR = \sim RR.$$

This **RR** is a fixed point for negation. It is neither true nor false. I do not leave logic behind. I imagine new states of logical discourse that are beyond the true and the false. Your set performs this transition to imaginary Boolean values.

Now Dr. HM says: Well I see you fellows are beginning to foment an argument. I feel that I must point out to you that logical paradox occurs only in the domain of language. There is no such matter as the paradox of the Russell set in the natural domain. In the natural domain, all apparent contradictions are only antimonies in the eyes of some obsever. Nature herself runs in the single valued logic of the evolutionary flow. This is why I emphasize that it is only in the linguistic domain of coordinations of coordinations that the

eigenforms arise. At the biological level there are processes that can be seen as recursions, but this seeing is already at the level of the coordinations. There is no mystery in this, but it is neccessary to round out the mathematical models with the prolific play and dynamics of the underlying biology.
In this sense biology is prior to physics as well as cognition.

At this point a tremour shakes the bar and the lights go out. I am sorry folks, the bartender says from the darkness, but this is another one of our natural events in the single valued logical flow of biological time -- a small earthquake. I will have to ask you to leave now for your own safety. And so the discussion ended, unfinished but perhaps that was for the best.

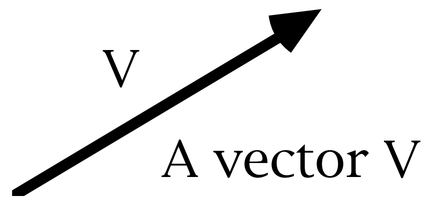
A Remark

The story in this section presents a number of different points of view about the cybernetics of fixed points. Fixed points can be produced by infinite recursion, by direct self-reference, through the linguistics of lambda calculus, and by approximation to inifinites. Mr. D is a fictionalized version of Douglas Harding the man who indeed realized that he did not have a head, and had the courage to write about it. The good Drs. at the bar represent these points of view and are thinly disguised representatives of the viewpoints of Heinz von Foerster, Alonzo Church and Haskell Curry (Dr. CC), Humberto Maturana and the physicists Julian Barbour. Charlie represents the American mathematical philosopher Charles Sanders Peirce. All this is only the beginning. The most famous fixed point of them all is the Universe herself, acted here by the bartender.

XV. Appendix - Eigenform, Eigenvalue and Quantum Mechanics

There are two reasons for including a discussion of quantum mechanics in this essay. On the one hand the quantum mechanics has been a powerful force in asking us to rethink our notions of objects and causality. On the other hand, von Foerster's notion of eigenform is an outgrowth of his background as a quantum physicist. We should ask what eigenforms might have to do with quantum theory and with the quantum world.

In this section we meet the concurrence of the view of object as token for eigenbehavior and the observation postulate of quantum mechanics. In quantum mechanics observation is modeled not by eigenform but by its mathematical relative the eigenvector. The reader should recall that a *vector* is a quantity with magnitude and direction, often pictured as an arrow in the plane or in three dimensional space.



In quantum physics [11], the state of a physical system is modeled by a vector in a high-dimensional space, called a Hilbert space. As time goes on the vector rotates in this high dimensional space. Observable quantities correspond to (linear) operators $\mathbf{H}$ on these vectors $\mathbf{v}$ that have the property that the application of $\mathbf{H}$ to $\mathbf{v}$ results in a new vector that is a multiple of $\mathbf{v}$ by a factor λ.
(An operator is said to be linear if $H(av + w) = aH(v) + H(w)$ for vectors v and w, and any number a. Linearity is usually a simplifying assumption in mathematical models, but it is an essential feature of quantum mechanics.)

In symbols this has the form

$$\mathbf{Hv} = \lambda\mathbf{v}.$$

One says that $\mathbf{v}$ is an *eigenvector* for the operator $\mathbf{H}$, and that λ is the *eigenvalue.* The constant λ is the quantity that is observed (for example the energy of an electron). These are particular properties of the mathematical context of quantum mechanics. The λ can be eliminated by replacing $\mathbf{H}$ by $\mathbf{G} = \mathbf{H}/\lambda$ (when λ is non zero) so that

$$\mathbf{Gv} = (\mathbf{H}/\lambda)\mathbf{v} = (\mathbf{Hv})/\lambda = \lambda\mathbf{v}/\mathbf{k} = \mathbf{v}.$$

Thus

$$\mathbf{Gv} = \mathbf{v}.$$

In quantum mechanics observation is founded on the production of eigenvectors v *with* $Gv=v$ *where* v *is a vector in a Hilbert space and* G *is a linear operator on that space.*

Many of the strange and fascinating properties of quantum mechanics emanate directly from this model of observation. In order to observe a quantum state, its vector is projected into an eigenvector for that particular mode of observation. By projecting the vector into that mode and not another, one manages to make the observation, but at the cost of losing information about the other possibilities inherent in the vector. This is the source, in the mathematical model, of the complementarities that allow exact determination of the position of a particle at the expense of nearly complete uncertainty about its momentum (or vice versa the determination of momentum at the expense of knowledge of the position).

Observation and quantum evolution (the determinate rotation of the state vector in the high dimensional Hilbert space) are interlocked.
Each observation discontinuously projects the state vector to an eigenvector. The intervals between observations allow the continuous evolution of the state vector. This tapestry of interaction of the continuous and the discrete is the basis for the quantum mechanical description of the world.

The theory of eigenforms is a sweeping generalization of quantum mechanics that creates a context for understanding the remarkable effectiveness of that theory. If indeed the world of objects is a world of tokens for eigenbehaviors, and if physics demands forms of observations that give numerical results, then a simplest example of such observation is the observable in the quantum mechanical model.

Is the quantum model, in its details, a consequence of general principles about systems? This is an exploration that needs to be made. We can only ask the question here. But the mysteries of the interpretation of quantum mechanics all hinge on an assumption of a world external to the quantum language. Thinking in terms of eigenform we can begin to look at how the physics of objects emerges from the model itself.

Where are the eigenforms in quantum physics?

They are in the mathematics itself. For example, we have the simplest wave-function

$$\varphi(x,t) = e^{i(kx - \omega t)}.$$

Since we know that the function $E(x) = e^x$ is an eigenform for orperation of differentiation with respect to x, $\varphi(x,t)$ is a special multiple eigenform from which the energy can be extracted by temporal differentiation, and the momentum can be extracted by spatial differentiation. We see in $\varphi(x,t)$ the complexity of an individual who presents many possible sides to the world. $\varphi(x,t)$ is an eigenform for more than one operator.
It is this internal complexity that is mirrored in the uncertainty relations of Heisenberg and the complementarity of Bohr. The eigenforms themselves, as wave-functions, are *inside* the mathematical model, on the *other side* of that which can be observed by the physicist.

We have seen eigenforms as the constructs of the observer, and in that sense they are on the side of the observer, even if the process that generates them is outside the realm of his perception. This suggests that we think again about the nature of the wave function in quantum mechanics. Is it also a construct of the observer?
To see quantum mechanics and the world in terms of eigenforms requires a turning around, a shift of perception where indeed we shall find that the distinction betweem model and reality has disappeared into the world of appearance.

This is a reversal of epistemology, a complete turning of the world upside down. Eigenform has tricked us into considering the world of our experience and finding that it is our world, generated by our actions and that it has become objective through the self-generated stabilities of those actions. We are brought into his world and we participate in the making of it. We can begin to see the genesis and tautological nature of quantum theory.

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