MIXED EXTENSION OF HYPERGAMES AND ITS APPLICATIONS TO INSPECTION GAMES

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ABSTRACT

In this paper, we extend hypergame models by introducing mixed strategies and illustrate that the mixed extension enables us to deal with hypergames with cardinal utilities, while the literature has dealt only with hypergames with ordinal utilities. We then show some unique features of mixed-strategy equilibria of hypergames (hyper Nash equilibrium [4: Kijima, 1996]) and study the comparative statics of equilibria due to change in misperceptions about cardinal utilities. Finally, we examine these findings in the framework of inspection games [1: Avenhaus et al., 1996].
Keywords: hypergame, hyper Nash equilibrium, mixed strategy, cardinal utility, inspection games.

1. INTRODUCTION

Hypergame theory [2] deals with agents who have misperceptions about the game in which they are involved.

Conventional hypergame model assumes only pure strategies as agents' strategies. Therefore, when we argue the shift of agents' behavior led by change in agent's misperceptions about utilities, we only have been able to analyze effect of 'drastic change' in misperceptions about the ordinal utilities. However, in reality, there are cases where such 'small change' in cardinal utilities that do not affect ordinal utilities may influence the agents' behavior. We want to construct a model that can deal with such cases.

To apply our analysis, we adopt inspection game, a model that deals with a situation in which the customs check illegal imports of travelers at an airport. This game has only one mixed-strategy Nash equilibrium (and no pure-strategy Nash equlibrium). It is said that the equilibrium explains the customs' random sampling well. We may intuitively suppose that the proportion of the random sampling is influenced not only by the customs' perception about travelers' ordinal utilities but also by their perception about travelers' cardinal utilities that do not affect ordinal utilities. We try to give a clear illustration about the situation in the application section.

2. CONVENTIONAL MODEL – HYPERGAME THEORY

Classical noncooperative games treat situations in which no agent has misperception.

Definition 2.1 (noncooperative games)

A noncooperative game is given by (N, S, u), where:
- $N = \{1, ..., n\}$ is a set of agents.
- $S = \times_{i \in N} S_i$ is a set of strategies, where S_i is a set of strategies of agent *i*.
- $u = (u_i)_{i \in N}$ is a profile of utility functions, where $u_i : S \to R$ is agent *i*'s utility function.

Nash equilibria are solution concepts of noncooperative games. An agent's strategy in a Nash equilibrium is called his/her Nash strategy.

Definition 2.2 (Nash equilibria)
$s^* = (s_i^*, s_{-i}^*) \in S$ is a Nash equilibrium of a noncoopereative game (N, S, u) iff $^\forall i \in N$, $^\forall s_i \in S_i$, $u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$.

On the other hand, hypergame theory [2] is a framework for dealing explicitly with agents who have misperceptions about the situations in which they are involved. Since it is almost impossible to understand the situation completely in the real world, this extension is quite natural so as to cover more realistic cases. Although this framework has been extended in several ways [2,3], we focus on simple hypergames because it is the most basic hypergame models.

Definition 2.3 (simple hypergames)
A simple hypergame is given by $(N, (G^i)_{i \in N})$, where:
- $N = \{1, ..., n\}$ is a set of agents involved in the situation
- $G^i = (N^i, S^i, u^i)$ is the subjective game of agent *i*, where :
  - N^i is a set of agents perceived by agent *i*.
  - $S^i = \times_{j \in N^i} S_j^i$ is a set of strategies perceived by agent *i*, where S_j^i is a set of strategies of agent *j* perceived by agent *i*.
  - $u^i = (u_j^i)_{j \in N^i}$ is a profile of utility functions perceived by agent *i*, where $u_j^i : S^i \to R$ is agent *j*'s utility function perceived by agent *i*.

In a simple hypergame, it is assumed that each agent perceives the situation subjectively in a form of a complete information game. We call a game perceived by an agent his/her subjective game. In general, an agent's subjective game may be different from another. We call the whole game, the list of the subjective games of all agents, hypergame.

Hyper Nash equilibrium [4] provides a solution concept of simple hypergames.

Definition 2.4 (hyper Nash equilibria)
$(s_i^{i*})_{i \in N} \in \times_{j \in N} S_j^j$ is a hyper Nash equilibrium of a hypergame iff $^\forall i \in N$, $s_i^{i*} \in N(G^i)_i$, where $N(G^i)_i$ is a set of agent *i*'s Nash strategies of G^i.

It is assumed that each agent plays according to a Nash strategy in his/her subjective game. A hyper Nash equilibrium is defined as a profile of such plays.

3. OUR MODEL

So far the literature has dealt only with hypergame models with pure strategies. We now introduce mixed strategies to hypergame framework, that is, we consider mixed extension of every subjective game.

Definition 3.1 (mixed extension of simple hypergames)
We call $(N, (G^i)_{i \in N})$ a mixed extension of hypergames iff $\forall i \in N$, G^i is a mixed extension.

Same as noncooperative games, we can define mixed-strategy Nash equilibria in every subjective game in the hypergame, which enables us to consider mixed-strategy hyper Nash equilibria.

Now, we discuss their unique features. Our first result is the following existence theorem.

Theorem 3.1 (existence theorem)
Every finite hypergame with mixed strategies has at least one hyper Nash equilibrium.

This is a natural generalization of Nash's theorem [5] about noncooperative games.

In the subsequent analysis, we analyze a specific class of hypergames in which agents misperceive the other's utilities, but perceive the other components of the situation correctly. Formally, $\forall i, j \in N$, $N^i = N$ and $S^i_j = S^j_j$. It is because of our motivation mentioned in the introduction, that is, to study comparative statics about changes in misperceptions of utilities. We call this class of hypergames 'perturbed hypergame situations'.

Before defining the class, we first introduce 'base games'. In this situation, the following complete information game is well-defined.

Definition 3.2 (base games)
A base game $G = (N, S, u)$ is a noncooperative game generated from a hypergame $(N, (G^i)_{i \in N})$ satisfying the following conditions:
- $S_i = S^i_i$
- $u_i = u^i_i$

A base game is a game where each agent's strategy set is his/her strategy set in his/her own subjective game in the hypergame, and the same goes for the utility functions. It can be regarded as a 'real game'.

A perturbed hypergame situation is a perturbation of a base game on misperceptions of utilities. Using the idea of base games, we define the class of perturbed hypergame situations as follows:

Definition 3.3 (perturbed hypergame situations)
A hypergame $(N, (G^i)_{i \in N})$ is a perturbed hypergame of base game $G = (N, S, u)$, iff we have $\forall i, j \in N$, $N^i = N$, $S_j^i = S_j$.

In a perturbed hypergame situation, all agents correctly perceive the set of agents involved in the situation and every agent's strategy set, while they may misperceive the others' utilities.

4. THE DERIVATION AND COMPARATIVE STATICS OF EQUILIBRIA

In this section, we first argue derivation procedure of mixed-strategy hyper Nash equilibria and then study the comparative statics of the equilibria. Although we can discuss more general cases in a similar way, we focus on 2x2 hypergames (from which a 2x2 base game can be generated) for simplicity here.

Let us see derivation procedure. At first, let us consider a base game shown below (Fig. 4.1) where there are two agents, say, 1 and 2. Suppose each agent has two strategies. Since we want to analyze mixed-strategy hyper Nash equilibria, we assume $a_3 > a_1$, $a_2 > a_4$, $b_1 > b_2$, $b_4 > b_3$ in this matrix. Under this condition, there is only one mixed-strategy Nash equilibrium and no pure-strategy Nash equilibrium.

1＼2	s_{21}	s_{22}
s_{11}	$a_1,\ b_1$	$a_2,\ b_2$
s_{12}	$a_3,\ b_3$	$a_4,\ b_4$

Fig. 4.1 a base game

In a perturbed hypergame situation, agents may misperceive the opponent's utilities. Let us suppose Fig. 4.2 show a subjective game of agent 1. We call it G^1. He, agent 1, misperceives agent 2's utilities and degree of the misperceptions is expressed as $\alpha_1 \sim \alpha_4$ in the matrix. Since we want to analyze the effect of small change in misperception about utilities that does not affect ordinal preferences, we assume that the order relations of agent 2's utilities remain the same as those in the base game, that is, we assume $b_1 + \alpha_1 > b_2 + \alpha_2$, $b_4 + \alpha_4 > b_3 + \alpha_3$.

1＼2	s_{21}	s_{22}
s_{11}	$a_1,\ b_1 + \alpha_1$	$a_2,\ b_2 + \alpha_2$
s_{12}	$a_3,\ b_3 + \alpha_3$	$a_4,\ b_4 + \alpha_4$

Fig. 4.2 agent 1's subjective game G^1

G^1 has no pure-strategy Nash equilibrium. We introduce mixed strategies into this game. Let $p^1, q^1 \in [0,1]$. p^1 is probability with which agent 1 thinks that he chooses his strategy s_{11}. It means he thinks he chooses his another strategy s_{12} with probability $1 - p^1$. Likewise, let q^1 be probability with which agent 1 thinks that agent 2 chooses her (agent 2's) strategy s_{21}.

Even when considering mixed strategies, there exists only one Nash equilibrium in G^1. The unique Nash equilibrium of this game is given by the intersection of each agent's best response graph (Fig. 4.3).

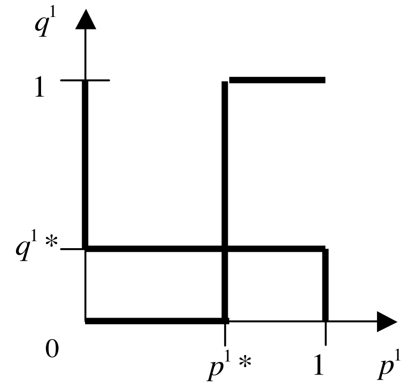

Fig. 4.3 Nash equilibrium in G^1

In a similar way, we can consider a subjective game of agent 2, say, G^2 (Fig. 4.4,) and its Nash equilibrium (Fig. 4.5). In G^2, $\beta_1 \sim \beta_4$ expresses agent 2's degree of misperceptions about agent 1's utilities. We also introduce mixed strategies into this game in the same way as G^1, that is, we assume that agent 2 thinks that agent 1 chooses his strategy s_{11} with probability p^2 and agent 2 chooses her strategy s_{21} with probability q^2. The unique Nash equilibrium of G^2 is given by the intersection of each agent's best response graph (Fig. 4.5).

1＼2	s_{21}	s_{22}
s_{11}	$a_1+\beta_1,\ b_1$	$a_2+\beta_2,\ b_2$
s_{12}	$a_3+\beta_3,\ b_3$	$a_4+\beta_4,\ b_4$

Fig. 4.4 agent 2's subjective game G^2

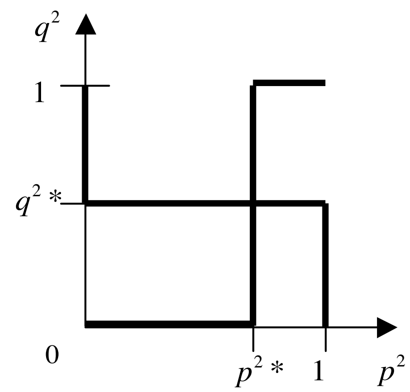

Fig. 4.5 Nash equilibrium in G^2

As mentioned previously, a hyper Nash equilibrium is a profile of each agent's Nash strategy in his/her own subjective game. In this hypergame, the unique hyper Nash equilibrium (p^*, q^*) is

$$p^* = p^1{}^* = \frac{-b_3 + b_4 - \alpha_3 + \alpha_4}{b_1 - b_2 - b_3 + b_4 + \alpha_1 - \alpha_2 - \alpha_3 + \alpha_4}$$

$$q^* = q^{2*} = \frac{-a_2 + a_4 - \beta_2 + \beta_4}{a_1 - a_2 - a_3 + a_4 + \beta_1 - \beta_2 - \beta_3 + \beta_4}$$

By examining the derivation of equilibria in two-agent hypergames, we have the following theorem, which we call derivation theorem.

Theorem 4.1 (derivation theorem)
In two-agent hypergames, if there exists a unique mixed- strategy hyper Nash equilibrium, the equilibrium coincides with the Nash equilibrium in $\tilde{G} = (N, S, (u_1^2, u_2^1))$.

$\tilde{G}$ is a complete information game where each agent's utilities are his/her utilities in the opponent's subjective game. It enables us to simplify calculation of the hyper Nash equilibrium.

Next, we study comparative statics of equilibria. In the literature, comparative statics has been carried out only with respect to drastic change in misperceptions about the ordinal utilities and the move of the equilibrium is not continuous.

In the previous example, the hyper Nash equilibrium (p^*, q^*) is a 'continuous' function of $\alpha_1 \sim \alpha_4$ and $\beta_1 \sim \beta_4$, and its derivative is not 0, where $\alpha_1 \sim \alpha_4$ and $\beta_1 \sim \beta_4$ meet the conditions set above. Thus, no matter how small the change of misperception is, it leads to the shift of the hyper Nash equilibrium. We cannot discuss this effect of small change in conventional models. Although this is restricted to cases of 2x2 hypergames here, it can be extended to more general results without difficulties.

Theorem 4.2 (continuity theorem)
Every mixed-strategy hyper Nash equilibrium moves continuously with continuous change in misperception.

5. APPLICATION TO INSPECTION GAMES

Now, we apply our model to situations called inspection games [1] and examine the intuitive implications regarding our analysis.

Consider a scene at an airport. A Traveler (T) tries to import wine and the Customs (C) checks if it is illegal or not. According to the law of this country, the small amount of import is not illegal, while the large amount of import is illegal. The more T imports wine, the better off he is because of the price gap between in his/her country and overseas. However, if T imports illegally and is busted by C's check, then wine is confiscated and T is fined. On the other hand, C's purpose is to block illegal imports primarily and to save the cost for the inspection. Hence, overlooking illegal imports without check is the worst result for C. If C knows T imports legally, C prefers not to check to save the cost.
This situation called inspection games can be formulated by the matrix below (Fig. 5.1). There are two agents, Traveler (T) and Customs (C). T's strategies are to import illegally (Illegal) and to import legally (Not illegal), while C's strategies

are to check (Check) and not to check (Not check). Their cardinal utilities reflect their gain or loss.

T＼C	Check	Not check
Illegal	$-35, -1$	$15, -10$
Not Illegal	$9, 0$	$9, 10$

Fig. 5.1 inspection game

We regard this game as a base game and analyze cases where the agents have misperceptions about the opponent's utilities (perturbed hypergame situation).

Let Fig. 5.2 be T's subjective game, G^{T}. In this game, T misperceives C's cost for the inspection, where α indicates misperception. If α is a negative number, that means T overrates C's cost for the inspection. Since we want to analyze effects of small change of misperception, we assume that the order relations of C's utilities remain the same as those in the base game, that is, we assume $-9 < \alpha < 10$.

T＼C	Check	Not check
Illegal	$-35, -1+\alpha$	$15, -10$
Not Illegal	$9, 0+\alpha$	$9, 10$

Fig. 5.2 T's subjective game G^{T}

Since G^{T} has no pure-strategy Nash equilibrium, we introduce mixed strategies into this game. We assume that T thinks that T chooses Illegal with probability p^{T} and C chooses Check with probability q^{T} ($p^{\mathrm{T}}, q^{\mathrm{T}} \in [0,1]$). Then, there exists one mixed-strategy Nash equilibrium in G^{T}; the unique Nash equilibrium of this game $(p^{\mathrm{T}}*, q^{\mathrm{T}}*)$ is

$$(p*^{\mathrm{T}}, q*^{\mathrm{T}}) = (\frac{10-\alpha}{19}, \frac{6}{50})$$

Similarly, let Fig. 5.3 be C's subjective game, G^{C}. In this game, C misperceives T's benefit of the illegal import when it is not checked by C. The misperception is indicated by β. If β is a negative number, that means C underestimates the benefit. We assume that the order relations of T's utilities remain the same here, too, that is, we assume $-6 < \beta$.

T＼C	Check	Not check
Illegal	$-35, -1$	$15+\beta, -10$
Not Illegal	$9, 0$	$9, 10$

Fig. 5.3 C's subjective game G^{C}

We introduce mixed strategies in the same way as G^{C}, that is, we assume that C thinks that T chooses Illegal with probability p^{C} and C chooses Check with probability q^{C} ($p^{\mathrm{C}}, q^{\mathrm{C}} \in [0,1]$). Then, G^{C} has one mixed- strategy Nash equilibrium; the unique Nash equilibrium of this game $(p^{\mathrm{C}}*, q^{\mathrm{C}}*)$ is

$$(p^{*C}, q^{*C}) = (\frac{10}{19}, \frac{6+\beta}{50+\beta})$$

As a result, we have the unique hyper Nash equilibrium (p^*, q^*).

$$(p^*, q^*) = (p^{*T}, q^{*C}) = (\frac{10-\alpha}{19}, \frac{6+\beta}{50+\beta})$$

We now examine some implications derived from the analysis above.

First, we can argue effects of misperceptions (difference from the base game) by this result. For example, if T overrates C's cost for the inspection ($\alpha < 0$) and C belittles T's benefit of the illegal import ($\beta < 0$), then T would increase Illegal and C would decrease Check compared with a case where they both have no misperceptions. This implication is compatible with our intuition; if you are a traveler and overrate the custom's cost for the inspection, you would anticipate that they would decrease the check and you would try to import illegally more.

Second, we have some implications from the ideas of the continuity theorem. As stated above, the hyper Nash equilibrium moves continuously with change in misperception. For example, if T's misperception α increases, that is, T estimates C's cost for the inspection less, then T's hyper Nash strategy changes from p^* to $p^{**}(< p^*)$, that is, T decreases the illegal import (Fig 5.4). As the theorem says, no matter how small the change of his misperception is, it leads to a shift of his hyper Nash strategy. It, in turn, leads to a shift of the hyper Nash equilibrium. The extent of the shift depends on how big the change of his misperception is.

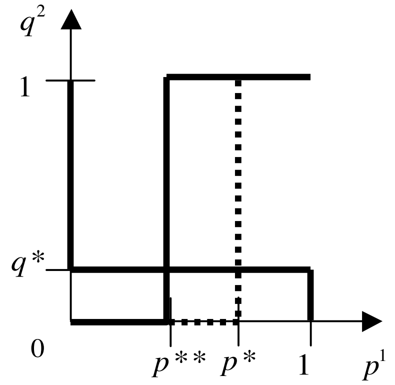

Fig. 5.4 shift of the hyper Nash equilibrium

The hyper Nash equilibrium changes from (p^*, q^*) to (p^{**}, q^*).

Using this feature of equilibria, we may refer to the possibility of ex-ante analysis to obtain the better future. In the inspection game, C wants T not to do illegal import, therefore, if possible, C wants to move the equilibrium in that direction. Now, based on the result shown in Fig. 5.4, we can advise C to make T estimate the cost for the inspection less, for example, to appeal their attitude to crack down on illegal imports strictly. It will lead to a shift of T's hyper Nash strategy and give the better future for C.

According to the derivation theorem, we can see the hyper Nash equilibrium coincides with the Nash equilibrium of a game $\tilde{G}$ (Fig. 5.5) which is generated from each subjective game.

T＼C	Check	Not check
Illegal	$-35, -1+\alpha$	$15+\beta, -10$
Not Illegal	$9, 0+\alpha$	$9, 10$

Fig. 5.5 $\tilde{G}$

6. CONCLUSIONS AND FURTHER RESEARCH

In this paper we introduced mixed strategies to hypergame models and showed that the mixed extension enables us to deal with hypergames with cardinal utilities. It implies that it enables us to explain about the effect of 'small change' in misperceptions about utilities, which was our main motivation of this research. We also examined these findings using inspection games. Our model can illustrate change of the proportion of both the traveler's behavior and the customs' random sampling.

Using the features of equilibria, we may conduct ex-ante analysis to obtain the better future. In terms of the inspection game, C wants T not to do illegal import. Therefore, if possible, C wants to move the equilibrium in that direction. Based on the result shown in Fig. 5.4, we can advise C to make T estimate the cost for the inspection less, for example, to appeal their attitude to crack down on illegal imports strictly. It will lead to a shift of T's hyper Nash strategy and give the better future for C. We believe that manipulating the opponent's misperceptions based on our model can provide an effective methodology for ex-ante analysis. Study of this methodology requires our future works. Furthermore, research on intuitive interpretation of the derivation theorem (Fig. 5.5) is also open.

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