Portfolio selection under multiple risk measures

Chunhui Xu, Jie Wang, Akiya Inoue
Dept. of Management Information Science
Chiba Institute of Technology
Chiba 275-0016, Japan
xchunhui@yahoo.co.jp

Abstract

The present paper considers portfolio selection problems when the investor's risk preferences are expressed with more than one risk measure, and proposes a method for solving optimization models for portfolio selection with multiple risk measures.

Portfolio selection experiments are conducted to show the effectiveness of the proposed model and solution method.

Key words: Portfolio selection, Risk, Return, Soft approach, Optimization, Value-at-Risk, Mean-Variance model.

1   Introduction

Portfolio selection has been modeled within the return-risk framework. While return is measured by the expected profit rate, there has been not a risk measure that is universally used.

Variance was suggested by Markowitz(1952) as a measure of risk, and due to its simplicity in computation, variance has been widely used in practical financial decisions. However, from the view of measuring risk, the variance is not a satisfactory measure of risk since it is a symmetric measure and penalizes gains and losses in the same way, and the variance is inappropriate to reflect the risk of low probability events.

To overcome these weak points, there proposed many risk measures, like the semi-variance, the absolute variance(Konno and Yamazaki(1991)), the value at risk(VaR, J.P. Morgan(1996)), the conditional value at risk(CVaR, Rockafellar and Uryasev(2000)) and the maximum loss(Young(1998)). Most of these risk measures are for measuring the downside risk, but some were proposed from computational considerations, see Dowd(2002) for an introduction to market risk measurement.

This paper takes this position about risk measure: risk is a subjective notion, different investors may use difference ways to express their understanding about risk, and it is reasonable that people may express their risk concern with more than one ways. Say, one may require that the variance should be less then that of a reference index or portfolio, and the VaR at 95% should not be bigger than 20%. Such a position raises the need for methods to portfolio selection under multiple risk measures.

Almost all of risk measures are nonlinear functions, which makes portfolio selection models computationally difficult to solve. With one risk measure, some smart methods were invented to change portfolio selection

models to ones that are easier to solve, such as a linear programming or quadratic programming. However, these techniques fail to work when there are multiple risk measures in a model, especially when VaR is included in a model.

The purpose of this paper is to propose a method to deal with this difficulty, which is based on the soft approach the author advocated for solving hard optimization models. To illustrate the advantages of the portfolio selected by our method, we do portfolio selection experiments using data from the New York stock market, and compare the risks and returns of the selected portfolio with that of market average.

The contents of this paper are arranged as follows: Section 2 formulates portfolio selection problems under multiple risk measures, Section 3 introduces a solution method based on the soft approach for complicated optimization, Section 4 does portfolio selection experiments using the proposed method and compares the results with one risk measure case, and Section 5 concludes this paper.

2 Optimization models for portfolio selection under multiple risk measures

This section formulates portfolio selection problems under multiple risk measures. We first introduce two risk measures which are popular in portfolio selection.

2.1 Two risk measures for portfolio selection

The present paper uses semi-variance and VaR as the risk measures in consider portfolio selection problems, because semi-variance is a representative in measuring downside risk and VaR is now a standard risk measure in financial industry.

Let n be the number of securities acceptable to the investor, x_i be the investment ratio on security i, then $x = (x_1, \cdots, x_n)$ is a portfolio.

Let the loss rate of portfolio x be $L(x)$, the VaR of portfolio x is the λ defined by the following formula,

$$inf\{\lambda | Pr\{L(x) \leq \lambda\} \geq \alpha\}, \tag{2.1}$$

where $Pr\{L(x) \leq \lambda\}$ is the probability that $L(x)$ will not be larger than λ, and $\alpha \in [0,1]$ is called the confidence level of VaR.

The VaR defined by (2.2) is the maximum loss rate such that the probability of losing more than a certain amount does not exceed $1 - \alpha$. For instance, when the VaR at a confidence level of 95% is 20%, the probability of losing more than 20% will not exceed 5%.

Because VaR is related to x and α, denote the VaR of portfolio x at confidence level α by $\lambda(x, \alpha)$, i.e.,

$$\lambda(x, \alpha) = inf\{\lambda | Pr\{L(x) \leq \lambda\} \geq \alpha\}, \tag{2.2}$$

Let $R(x)$ be the profit rate of portfolio x, which is uncertain at the time of planning. Suppose that we have a number of scenarios for the return of each security, let them be $r_{1t}, r_{2t}, \cdots, r_{nt}, t = 1, \cdots, d(T)$, where $d(T)$ is the number of scenarios.

Denote the average return of security j in these scenarios by $\bar{r}_j$, and that of portfolio x by $\bar{r}(x)$, that is,

$$\bar{r}_j = \frac{1}{d(T)} \Sigma_{t=1}^{d(T)} r_{it}, j = 1, 2, \cdots, n. \tag{2.3}$$

$$\bar{r}(x) = \Sigma_{j=1}^{n} \bar{r}_j x_j \tag{2.4}$$

Then the semi-variance of the return of portfolio x defined as follows is usually used as a measure of downside risk of the portfolio.

$$SVar(x) = \frac{1}{d(T)} \Sigma_{t=1}^{d(T)} |r_t(x) - \bar{r}(x)|_{-}^2 \tag{2.5}$$

where $r_t(x)$ is the return of portfolio x in scenario t, i.e., $r_t(x) = \Sigma_{j=1}^{n} r_{jt} x_j$.

2.2 Models for portfolio selection

Consider the following portfolio selection problem:

Select a portfolio for an investor, who cares the return and risk at the end of a plan period.

Suppose that short selling is not allowed, then selecting a portfolio is equal to choosing a point from the following set,

$$X = \{x | \Sigma_{i=1}^{n} x_i = 1, x_i \geq 0, i = 1, \cdots, n\}. \tag{2.6}$$

Since the investor cares return and risk, we consider two investment styles according to the criterion the investor prefers to emphasize, and build the corresponding portfolio selection model.

While portfolio selection under return focusing investment style selects a portfolio by maximizing the return subject to the restriction that the risk should not exceed some maximal level, which can be formulated as

$$Max_{x \in X} \quad r(x) : \lambda(x, \alpha) \leq \lambda_0, SVar(x) \leq \beta \tag{2.7}$$

Portfolio selection under risk focusing investment style selects a portfolio by minimizing the risk subject to the restriction that the return should exceed some minimal level, which can be formulated with the following optimization model,

$$Min_{x \in X} \quad \lambda(x, \alpha) : SVar(x) \leq \beta, r(x) \geq \mu \tag{2.8}$$

where μ is the minimal return level required, and β is the maximal risk in $SVar$ that is acceptable.

or

$$Min_{x \in X} \quad SVar(x) : \lambda(x, \alpha) \leq \lambda_0, r(x) \geq \mu \tag{2.9}$$

where λ_0 is the maximal risk level in VaR that is acceptable.

Parameters α, β, λ_0 and μ in these models are called control parameters in portfolio selection, which are to be set by the investor's preferences and requirements for investment.

Since VaR is generally a nonconvex and nonsmooth function, and SVar is a nonsmooth function, conventional optimization methods are helpless in solving these models. The next section will use the soft approach we advocated in recent years to solve these models.

3   A solution method for portfolio selection models

The soft approach was proposed for solving complicated optimization models when seeking an optimal solution is theoretically or practically impossible, see Xu(2003), and Xu and Ng(2006) for details about this approach.

This section first outlines the solution process of the soft approach, and then introduces corresponding algorithms for solving the optimization models built in previous section.

3.1   Solution process of the soft approach

The soft approach produces a solution through the following two stages.

- Stage 1: Sample the feasible set to generate a finite subset S.

  Let G be a set of good enough solutions, the soft approach requires the probability with which S contains at least one good enough solution to be high, i.e., the soft approach requires the following condition to be satisfied:

$$Pr\{|S \cap G| \geq 1\} \geq q\% \tag{3.1}$$

  Where q is a number which is taken as a high probability. Consequently, the best sample in S will be a good enough solution with a probability not smaller than $q\%$.

- Stage 2: Select the best sample from the sample set S.

  Calculate the performance of each sample, then the sample with the highest performance is the solution the soft approach produces.

In order to check if a sample set generated satisfies condition (3.1), we need to define the good enough set properly and check how samples are picked.

The soft approach uses order instead of value in defining the notion of good enough solution, for instance, the top-$k\%$ solutions may be taken as G. The soft approach suggests to use uniform sampling methods in picking samples if there is no information about the distribution of the good enough solutions. In this way, the soft approach can quantify the quality of the solution obtained.

For instance, when the good enough solutions are defined as the top 1% solutions, and $1,000$ samples are taken from the feasible set by some uniform sampling method, then the probability that none of the good enough solutions is included in the sample set is $(1 - 1\%)^{1,000} < 0.1\%$, consequently, the probability that the sample set contains at least one good enough solution is bigger than 99.9%.

In other words, if we take 1,000 uniform samples from the feasible set and selecting the best sample as the final solution, then the final solution is highly likely (with a probability larger than 99.9%) a top 1% solution.

To focus on the main issue, the present paper assumes that getting a top 1% solution with a probability higher than 99.9% is acceptable in solving the models built in previous section. Consequently, we only need 1,000 uniform samples in Stage 1.

3.2 An algorithm for solving portfolio selection models

Because the solution process of model (2.7) is similar with that of models (2.8) and (2.9), we use (2.7) to illustrate the algorithm.

Rewrite model (2.7) as follows,

$$Max_{x \in X_1^f} \quad r(x) \tag{3.2}$$

where X_1^f is given by

$$X_1^f = \{x \in X | \lambda(x,\alpha) \leq \lambda_0, SVar(x) \leq \beta\}$$

The soft method solves this model following the two-stage process.

- Stage 1: Take 1,000 uniform samples from X_1^f.

There are several methods for taking uniform samples from X_1^f, here we suggest to sample X_1^f as follows.

1. generate a uniform sample of X.

   It is easy to generate a number which is uniformly distributed over interval $[0,1]$, let $y_1, \cdots, y_n$ be n uniform numbers over $[0,1]$, then $(\frac{y_1}{\Sigma_{i=1}^n y_i}, \cdots, \frac{y_n}{\Sigma_{i=1}^n y_i})$ is a uniform sample of X.

2. check the feasibility of the sample generated.

   The generated sample is a feasible point of X_1^f if constraints $\lambda(x,\alpha) \leq \lambda_0$ and $SVar(x) \leq \beta$ are satisfied.

   Keep the sample if it is a feasible point in X_1^f, reject it otherwise.

Repeat this generating and checking process till 1,000 feasible samples of X_1^f are obtained. Denote the set of generated uniform samples of X_1^f by S_1.

- Stage 2: Choose the sample from S_1 with the highest return.

To find out the sample with the highest return is easy, which can be done by evaluating the return of each sample in S_1 and choosing the one with the biggest return.

There are several methods for evaluating VaR, such as the historical simulation method, the delta-normal method and the Monte Carlo method, refer to Dowd(2002) for details about these methods. Here we suggest to use the historical simulation method for estimating VaR because this method only uses historical data, no other assumptions about future market are imposed.

4 Portfolio selection experiments

This section tests the proposed soft method with investment experiments using real data from the US stock market.

4.1 Settings of Experiments

The experiments are conducted in the following conditions.

- Securities for investment: the 30 component securities of the DOW JONES index,

- Beginning of T: the first trade day in year 2007, denoted by t_0.

- End of T: the first trade day of each month in year 2008, denoted by t_T.

  We do one experiment for each of the following 12 plan periods.

T	Length of T ($t_0 \sim t_T$)	T	Length of $T(t_0 \sim t_T)$
T_1	12 months: $t_0 \sim 2008.1.2$	T_7	18 months: $t_0 \sim 2008.7.1$
T_2	13 months: $t_0 \sim 2008.2.1$	T_8	19 months: $t_0 \sim 2008.8.1$
T_3	14 months: $t_0 \sim 2008.3.1$	T_9	20 months: $t_0 \sim 2008.9.1$
T_4	15 months: $t_0 \sim 2008.4.1$	T_{10}	21 months: $t_0 \sim 2008.10.1$
T_5	16 months: $t_0 \sim 2008.5.1$	T_{11}	22 months: $t_0 \sim 2008.11.1$
T_6	17 months: $t_0 \sim 2008.6.1$	T_{12}	23 months: $t_0 \sim 2008.12.1$

- Historical price data used: adjusted daily close prices of the 30 securities from year 2003 to year 2006 [1].

  We use price data of four years right before the plan period.

4.2 Scenario generation and parameter determination

For each plan period, we generate return scenarios by using the price data of the securities in the past. [2].

Let $m(T)$ be the number of business days in a plan period T, and m be the number of price data to be used in scenario generation, then we generate $d(T) = m - m(T)$ scenarios for the return of each security by the following formula,

$$r_{it} = \frac{P_j(t + m(T)) - P_j(t)}{P_j(t)}, t = 1, 2, \cdots, d(T), i =, 2, \cdots, 30. \tag{4.1}$$

Parameter α is set to 95% in the twelve experiments, a regular value for the confidence level in using VaR.

Parameter λ_0 is set to 10% in all experiments, which means that we will control the VaR at confidence level 95% within 10%.

Assigning a value to parameter β is not so easy because its economic explanation is not clear. Here we set β as the semi-variance of the market average, which means that we will control the SVar less than the market average.

4.3 Results of Experiments

For each of the plan periods, we select a portfolio by solving model (2.7) with the proposed two-stage method.

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[1] All data used are available in Yahoo! Finance's website.

[2] For the convenience of processing data and programming, we take 20 trade days as one month in the experiments, that is, $T_1 = 1 * 20$ trade days, $T_2 = 2 * 20$ trade days, and so on.

To evaluate the portfolios selected by the soft method, we consider a special portfolio, which allocates the initial investment fund to each security with the same ratio, denote it by x^0, i.e.,

$$x^0 = (\frac{1}{30}, \cdots, \frac{1}{30}),$$

and name x^0 the neutral portfolio. We take the return and risk of x^0 as the market average return and market average risk, respectively.

For each plan period, we calculate the return and risk of the selected portfolio. The return data of the selected portfolios and x^0 are summarized in Table 1, the VaR in Table 2 and SVar in Table 3 below.

Return(%)	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}	T_{11}	T_{12}
$r_{\text{p}}(x^*)$	16.78	18.00	19.11	20.07	20.67	21.25	22.13	23.92	26.60	28.88	30.18	29.63
$r_{\text{p}}(x^0)$	12.41	12.97	13.48	14.00	14.64	15.47	16.24	17.16	18.16	19.07	20.21	21.25

Table 1: Return comparison between the selected portfolios and the market average

Risk (%)	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}	T_{11}	T_{12}
$\lambda(x^*, 95\%)$	-3.91	-4.20	-0.83	-5.65	-3.03	-6.95	-1.66	-4.80	-5.42	-2.41	-6.43	-7.57
$\lambda(x^0, 95\%)$	-2.44	-2.19	-2.77	-2.38	-2.15	-3.77	-3.13	-3.68	-4.30	-4.31	-6.06	-8.03

Table 2: VaR comparison between the selected portfolios and the market average

Risk (%)	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}	T_{11}	T_{12}
$SVar(x^*)$	0.26	0.21	0.25	0.25	0.30	0.23	0.29	0.35	0.37	0.36	0.31	0.34
$SVar(x^0)$	0.29	0.29	0.29	0.30	0.32	0.33	0.36	0.37	0.38	0.37	0.37	0.37

Table 3: SVar comparison between the selected portfolios and the market average

We plot Table 1 in Figure 1, and Tables 2 and 3 in Figure 2 for seeing the differences clearly.

Figure 1 shows clearly that selected portfolios produce a higher return than the market average in all plan periods.

Figure 2 shows that risk measured by VaR of the selected portfolios have been controlled within 10%, and the risk measured by SVar is lower than the market average in all plan periods.

Thus we confirmed that the proposed models and solution method provide an effective way to do portfolio selection when investors use several measures for risk.

5 Conclusions

This paper formulated portfolio selection problems wherein investors may use different measures for investment risk, and proposed a solution method for solving such portfolio selection models.

Portfolio selection experiments showed that the proposed method is effective in solving these models, thus the proposed method provides a usable method for solving complicated portfolio selection problems.
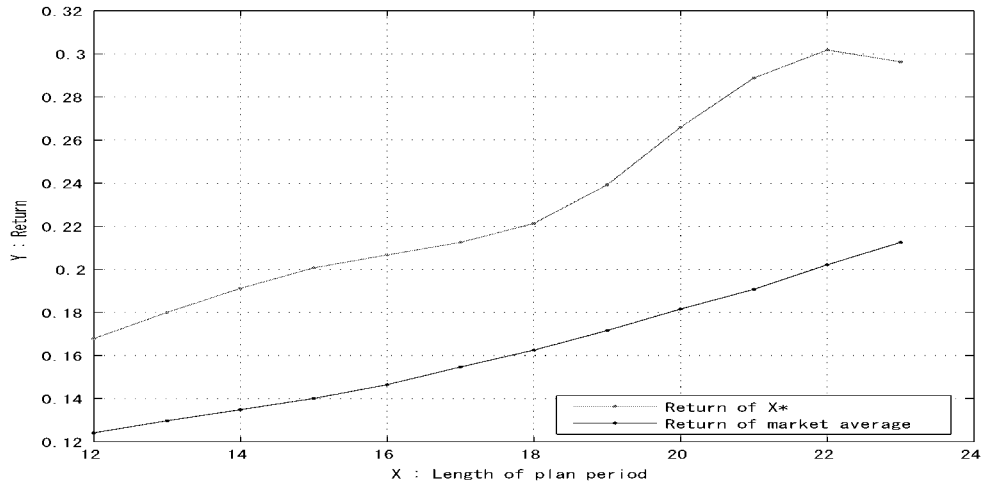
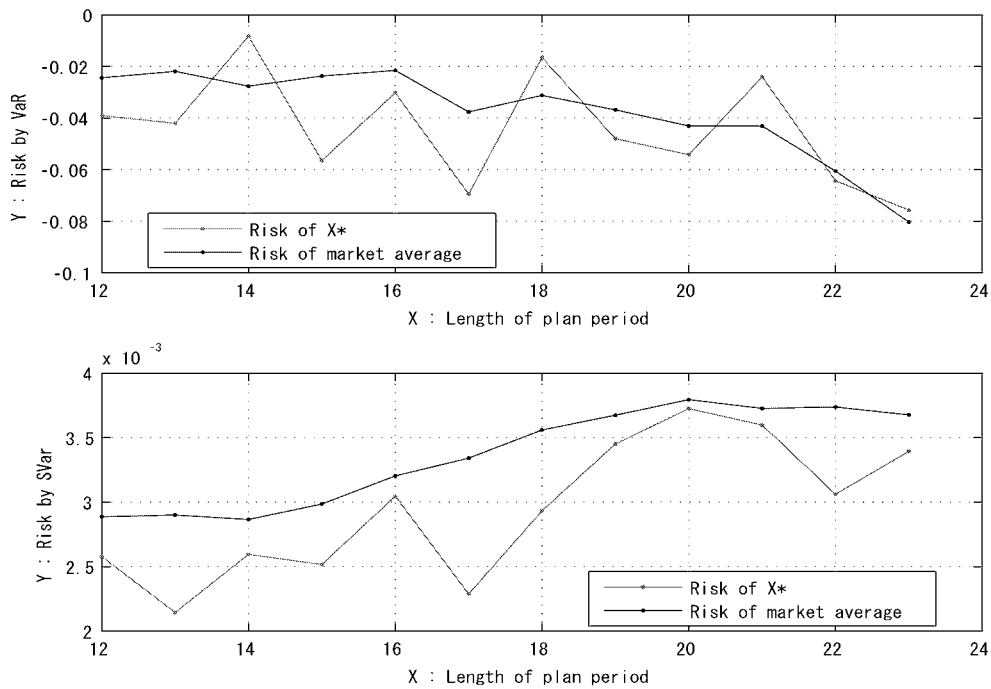Figure 1: Return comparisons between selected portfolios and market average

Figure 2: Risk comparisons between selected portfolios and market average

Acknowledgement

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